Final take-home exam

Exercise 1 Let $\Gamma = \mathbb{SL}_2(\mathbb{Z})$

1. Prove that the subspaces $M_k(\Gamma)$ (for various integers k) are in direct sum inside $\mathscr{O}(\mathscr{H})$ (the space of holomorphic functions on the upper half-plane $\mathscr{H}$).

2. Prove that for k even
$$\dim S_k(\Gamma) = \max(0, \dim M_k(\Gamma) - 1)$$
and that $f \to f\Delta$ induces an isomorphism $M_k(\Gamma) \simeq S_{k+12}(\Gamma)$.

3. Conclude that $(E_4^a E_6^b)_{4a+6b=k,a,b\geq 0}$ is a basis of $M_k(\Gamma)$.

4. Prove that there is an isomorphism of $\mathbb{C}$-algebras
$$\bigoplus_{k\in\mathbb{Z}} M_k(\Gamma) \simeq \mathbb{C}[X,Y].$$

5. Without doing all the nasty computations, explain how to prove that
$$\Delta = \frac{E_4^3 - E_6^2}{1728}$$
and
$$E_{12} - E_6^2 = \frac{2^6 \cdot 3^5 \cdot 7^2}{691}\Delta.$$

6. Deduce that if $\Delta = \sum_{n\geq 1} \tau(n) q^n$, then
$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

Exercise 2

1. Prove that there is a natural homeomorphism
$$\mathbb{SL}_2(\mathbb{R})/\mathbb{SO}_2(\mathbf{R}) \simeq \mathscr{H} := \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}.$$

2. Prove that the natural representation of $\mathbf{R}$ on $L^2(\mathbf{R})$ has no irreducible subrepresentation.

3. Give an example of a closed subgroup G of $\mathbb{GL}_n(\mathbf{R})$ and of a representation $V \in \operatorname{Rep}(G)$ with the property that there is a $\mathfrak{g} = \operatorname{Lie}(G)$-stable subspace $W \subset V^\infty$ whose closure is not stable under G.

Exercise 3 Let $\Gamma = \mathbb{SL}_2(\mathbb{Z})$ and let $f = \sum_{n\geq 1} a_n q^n \in S_k(\Gamma)$ be such that $a_1 = 1$ and f is an eigenvector of the Hecke operator T_n for all n. For each prime p let $\alpha_p, \beta_p \in \mathbb{C}$ be such that

$$1 - a_p X + p^{k-1} X^2 = (1 - \alpha_p X)(1 - \beta_p X).$$

Define

$$L(s) = \prod_p \frac{1}{(1 - \alpha_p^2 p^{-s})(1 - \alpha_p \beta_p p^{-s})(1 - \beta_p^2 p^{-s})}.$$

1. Briefly recall why $a_{mn} = a_m a_n$ for $\gcd(m, n) = 1$.

2. Prove that $a_{p^n} = \frac{\alpha_p^{n+1} - \beta_p^{n+1}}{\alpha_p - \beta_p}$ for all primes p and all $n \geq 1$.

3. Prove that a_n is a real number for any n.

4. Let

   $$A(s) = \sum_{n\geq 1} \frac{a_n^2}{n^s}.$$

   Prove that for $\mathrm{Re}(s)$ large enough we have

   $$A(s) = \frac{\zeta(s - k + 1)}{\zeta(2s - 2k + 2)} L(s).$$

5. Deduce that $s \to L(s)$ has meromorphic continuation to $\mathbb{C}$ and that there is a constant $c_k > 0$ depending only on k such that

   $$L(k) = c_k \langle f, f \rangle,$$

   where $\langle ., . \rangle$ is the Petersson inner product.

6. Let

   $$\Lambda(s) = \pi^{-3s/2} \Gamma(\frac{s}{2}) \Gamma(\frac{s+1}{2}) \Gamma(\frac{s-k+2}{2}) L(s).$$

   Prove that $\Lambda(2k - 1 - s) = \Lambda(s)$.

7. Let

   $$B(s) = \sum_{n\geq 1} \frac{a_{n^2}}{n^s}.$$

   Prove that

   $$L(s) = \zeta(2s - 2k + 2) B(s)$$

   and that $s \to B(s)$ has meromorphic continuation to $\mathbb{C}$.

Exercise 4 Let V be a unitary representation of $G = \mathbb{SL}_2(\mathbb{R})$ on a Hilbert space. Write $a(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ for $t > 0$ and $u(s) = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$.

1. Let $v \in V$ and let $g, g_n, h_n, h'_n \in G$ be such that $\lim_{n\to\infty} g_n = g$, $h_n.v = v = h'_n.v$ for all n, and $\lim_{n\to\infty} h_n g_n h'_n = 1$. Prove that $g.v = v$.

2. Suppose that $a(t).v = v$ for some $t > 1$. Prove that $u(s).v = v$ for all s.

3. Suppose that $u(s).v = v$ for some $s \neq 0$. Prove that $a(t).v = v$ for all $t > 0$. Deduce that v is fixed by G.

   Suppose now that $V^G = 0$. We want to prove that for all $v, w \in V$ we have $\lim_{g\to\infty}\langle g.v, w\rangle = 0$. We argue by contradiction and assume that this does not hold.

4. Prove that we can find $v', w' \in V$, $\alpha \in \mathbf{C}^*$, $t_n \to \infty$ and $v_0 \in V$ such that $\lim_{n\to\infty}\langle a(t_n).v', w'\rangle = \alpha$ and $\langle a(t_n).v', x\rangle \to \langle v_0, x\rangle$ for all $x \in V$.

5. Prove that $v_0 \neq 0$ and that v_0 is fixed by $u(1)$. Conclude.

   Let Γ be a lattice in G and let dx be the unique G-invariant probability measure on $X := \Gamma\backslash G$. Let $V = \{f \in L^2(X) |\ \int_{\Gamma\backslash G} f(x)dx = 0\}$, with the action of G given by $g.f(x) = f(xg)$ (for the natural action of G on X).

6. Prove that $V^G = 0$. Deduce that for all $\phi, \psi \in L^2(X)$ we have

$$\lim_{g\to\infty}\int_X \phi(x)\psi(xg)dx = \frac{1}{\mathrm{vol}(X)}\int_X \phi(x)dx \cdot \int_X \psi(x)dx.$$

7. (challenging) Prove that the translates Yg of the K-orbit $Y = (\Gamma \cap K)\backslash K$ in X become equidistributed on X as $g \to \infty$, i.e. whenever $f \in C_c(X)$ we have

$$\lim_{g\to\infty}\int_{Yg} f(y)dy = \int_X f(x)dx,$$

   where we endow Y with the unique K-invariant probability measure.