p-ADIC ÉTALE COHOMOLOGY OF PERIOD DOMAINS

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Abstract. We compute the p-torsion and p-adic étale cohomologies with compact support of period domains over local fields in the case of basic isocrystals for quasi-split reductive groups. As in the cases of ℓ-torsion or ℓ-adic coefficients, $\ell \neq p$, considered by Orlik, the results involve generalized Steinberg representations.

For the p-torsion case, we follow the method used by Orlik in his computations of the ℓ-torsion étale cohomology using as a key new ingredient the computation of Ext groups between mod p generalized Steinberg representations of p-adic groups. For the p-adic case, we don't use Huber's definition of étale cohomology with compact support as Orlik did since it seems to give spaces that are much too big; instead we use continuous étale cohomology with compact support.

Contents

1. Introduction

Let p be a prime number. One of the main results of [12] and [13] is the computation of the geometric p-adic étale cohomology of Drinfeld p-adic symmetric spaces in arbitrary dimension. The final result is analogous to the one in the case of ℓ-adic étale cohomology with $\ell \neq p$, which was known by the work of Schneider and Stuhler [62]. The Drinfeld symmetric spaces are among the most classical examples of *p-adic period domains* but it is well-known that they are very special[1]. In fact, the proofs in [12] and [13] use these unique properties of Drinfeld spaces hence it was not clear to us whether the results of loc. cit. would extend to more general p-adic period domains.

The purpose of this paper is to show that, for *compactly supported* p-torsion étale cohomology, it is possible to treat fairly general p-adic period domains. Moreover, the result is similar to the one for ℓ-torsion, $\ell \neq p$, cohomology with compact support obtained[2] by Dat [16] (for the Drinfeld spaces) and by Orlik [51] (in general). That this is the case is a little surprising since, as we will explain below, the p-adic étale cohomology with compact support (in the sense of Huber [37]) of p-adic

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[1] See [57, Sec. 3] for a list of such properties.

[2] The Euler characteristic of period domains was known before, thanks to Kottwitz and Rapoport, see [17] for a beautiful presentation.

period domains is not at all similar to its ℓ-adic counterpart, $\ell \neq p$, computed by Orlik [53], and seems to produce not very useful objects. On the other hand, the continuous compactly supported cohomology that we define gives reasonable objects (at least in the situation we consider or in the case of the complement of a subvariety in a proper analytic variety as considered[3] in [45]).

While the arguments in [12] and [13] are based on p-adic Hodge theory (via the syntomic method) and its integral versions [4, 5, 9], this paper combines a beautiful geometric construction due to Orlik [51] with a vanishing result for extensions between mod p representations of p-adic reductive groups. The proof of the second result is the main difference with the $\ell \neq p$ case. We are not able to recover the results of [12] and [13] using the methods used here, and conversely the methods in loc. cit. do not seem to give the results obtained in this paper for Drinfeld spaces (Poincaré duality with p-torsion coefficients holds for "almost proper" analytic varieties [45], but probably does not hold for general analytic varieties, at least in a naive sense). Orlik did recover in [54] the computation of p-adic pro-étale cohomology of Drinfeld spaces from [12] using his method – which is the one of this paper as well – but one encounters considerable technical difficulties[4] when working with the étale cohomology instead of the pro-étale one.

1.1. **Notation.** In order to state the main results of this paper we need to introduce some notation. Let C be the completion of an algebraic closure of $\mathbf{Q}_p$ and let $(G,[b],\{\mu\})$ be a *local Shtuka datum* over $\mathbf{Q}_p$. Here G is a connected reductive group over $\mathbf{Q}_p$, $[b]$ is an element of the Kottwitz set $B(G)$ of σ-conjugacy classes in[5] $G(\breve{\mathbf{Q}}_p)$, i.e., an isomorphism class N_b of isocrystals with G-structure over $\breve{\mathbf{Q}}_p$, and $\{\mu\}$ is a conjugacy class of geometric cocharacters of G. Moreover, we ask that $[b]$ lies in the Kottwitz set[6] $B(G,\mu)$, a certain finite subset of $B(G)$ defined roughly by a comparison between the Hodge polygon attached to μ and the Newton polygon attached to N_b.

The pair $(G,\{\mu\})$ gives rise to a generalized flag variety[7] $\mathscr{F} = \mathscr{F}(G,\{\mu\})$ defined over the field of definition E of $\{\mu\}$, a finite extension of $\mathbf{Q}_p$, local analogue of the reflex field in the theory of Shimura varieties. We will consider $\mathscr{F}$ as an adic space over $\mathrm{Spa}(E,\mathscr{O}_E)$. Letting $\breve{E} = E\breve{\mathbf{Q}}_p$, the *p-adic period domain* introduced by Rapoport and Zink [60]

$$\mathscr{F}^{\mathrm{wa}} = \mathscr{F}^{\mathrm{wa}}(G,[b],\{\mu\})$$

is a partially proper open subset of $\mathscr{F} \otimes_E \breve{E}$, classifying the weakly admissible filtrations of type $\{\mu\}$ on the isocrystal N_b. Basic examples of p-adic period domains are the adic affine spaces, the projective spaces, and the Drinfeld symmetric spaces (complements of the union of all $\mathbf{Q}_p$-rational hyperplanes in the projective spaces).

As we have already mentioned, Orlik computed in [51] the ℓ-adic compactly supported étale cohomology of these period domains when G is quasi-split over $\mathbf{Q}_p$, $[b]$ is a basic class, and $\ell \neq p$ is a sufficiently generic prime number. We will also assume that G is quasi-split over $\mathbf{Q}_p$ and that $b \in G(\breve{\mathbf{Q}}_p)$ is basic and s-decent[8]. We refer the reader to the main body of the article for these notions, introduced by Kottwitz (for the first one) and Rapoport-Zink (for the second one). This implies, for instance, that $b \in G(\mathbf{Q}_{p^s})$ and that the period domain $\mathscr{F}^{\mathrm{wa}} = \mathscr{F}^{\mathrm{wa}}(G,[b],\{\mu\})$ has a

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[3] In both cases this continuous compactly supported cohomology coincides with the naive one.

[4] For example, the rational p-adic pro-étale cohomology of an open ball has a simple description in terms of differential forms [14], but the integrality conditions coming from the p-adic étale cohomology make the computations subtler.

[5] $\breve{\mathbf{Q}}_p$ is the completion of the maximal unramified extension of $\mathbf{Q}_p$ in C.

[6] For the main result of the paper it would be enough to assume that $[b]$ belongs to the larger set $A(G,\mu)$, since all we need is that the period domain is nonempty, which is equivalent to $[b] \in A(G,\mu)$ by a result of Fontaine and Rapoport [25].

[7] If G is quasi-split over $\mathbf{Q}_p$, which will be the case in our main result, we can choose $\mu \in \{\mu\}$ defined over E and then $\mathscr{F} = \mathscr{F}(G,\{\mu\})$ is the quotient of G_E by the parabolic subgroup $P(\mu)$ associated to μ.

[8] The hypothesis that b is decent is harmless, since any σ-conjugacy class in $G(\breve{\mathbf{Q}}_p)$ contains an s-decent element for some positive integer $s \geq 1$.

canonical model (still denoted $\mathscr{F}^{\mathrm{wa}}$) over $E_s = E\mathbf{Q}_{p^s} \subset \overline{\mathbf{Q}}_p$. Let J_b be the automorphism group of N_b. It is a connected reductive group over $\mathbf{Q}_p$, which is an inner form of G (this is equivalent to b being basic). The natural action of $G(\breve{\mathbf{Q}}_p)$ on the flag variety $\mathscr{F} \otimes_E \breve{E}$ induces an action of $J_b(\mathbf{Q}_p)$ on the period domain $\mathscr{F}^{\mathrm{wa}}$. In particular, we obtain an action of $J_b(\mathbf{Q}_p) \times \mathscr{G}_{E_s}$, $\mathscr{G}_{E_s} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/E_s)$, on $H^*_{\text{ét},c}(\mathscr{F}^{\mathrm{wa}}_C, \mathbf{Z}/\ell^n)$ and $H^*_{\text{ét},c}(\mathscr{F}^{\mathrm{wa}}_C, \mathbf{Z}_\ell)$, for any prime ℓ. The main theorem of this paper gives a simple description of these representations in the case $\ell = p$.

Let T be a maximal torus of G such that μ factors through T, and let $W = N(T)/T$ be the (absolute) Weyl group of G with respect to T, which acts naturally on $X_*(T)$. Let W^μ be the set of Kostant representatives with respect to $W/\mathrm{Stab}(\mu)$, i.e., the representatives of shortest length in their cosets. The group $\mathscr{G}_{E_s}$ acts on W and preserves W^μ since μ is defined over E_s. One can associate to each $\mathscr{G}_{E_s}$-orbit $[w] \in W^\mu/\mathscr{G}_{E_s}$ the following objects:

- An integer $l_{[w]}$, the length of any element of $[w]$.
- For any prime ℓ, a $\mathbf{Z}/\ell^n[\mathscr{G}_{E_s}]$-module $\rho_{[w]}(\mathbf{Z}/\ell^n)$, which is simply the $\mathbf{Z}/\ell^n$-module of $\mathbf{Z}/\ell^n$-valued functions on $[w]$, with the obvious $\mathscr{G}_{E_s}$-action twisted (à la Tate) by $-l_{[w]}$.

We will simply write J instead of J_b from now on. Choose a maximal $\mathbf{Q}_p$-split torus S of J_{der} contained in T and a minimal $\mathbf{Q}_p$-parabolic subgroup P_0 of J containing S. Let $\Delta \subset X^*(S)$ be the associated set of relative simple roots. For each subset I of Δ, we denote by P_I the corresponding standard $\mathbf{Q}_p$-parabolic subgroup of J, so that $P_\varnothing = P_0$ and $P_\Delta = J$. Consider the compact p-adic manifold

$$X_I = J(\mathbf{Q}_p)/P_I(\mathbf{Q}_p).$$

If R is an abelian group, let

$$v^J_{P_I}(R) = i^J_{P_I}(R)/\sum_{I \subsetneq I'} i^J_{P_{I'}}(R), \quad i^J_{P_I}(R) = \mathrm{LC}(X_I, R),$$

be the corresponding generalized Steinberg representation of $J(\mathbf{Q}_p)$, with coefficients in R (here $\mathrm{LC}(?, R)$ is the space of locally constant functions on ? with values in R).

Finally, choose an invariant inner product $(-,-)$ on G, i.e., an inner product on $X_*(T') \otimes \mathbf{Q}$, for all maximal tori T' of G, compatible with the adjoint action of $G(\overline{\mathbf{Q}}_p)$ and the natural action of $\mathscr{G}_{\mathbf{Q}_p}$ on maximal tori of G. It induces an invariant inner product on J as well. For each $\mathscr{G}_{E_s}$-orbit $[w]$ of $w \in W^\mu$, define

$$I_{[w]} = \{\alpha \in \Delta \mid (w\mu - \nu, \omega_\alpha) \leq 0\}, \quad P_{[w]} := P_{I_{[w]}},$$

where $\omega_\alpha \in X_*(S) \otimes \mathbf{Q}$, $\alpha \in \Delta$, form the dual basis of Δ.

1.2. The main result.

Recall that, for $\ell \neq p$, we have the following computation of Orlik.

Theorem 1.1 (Orlik, [51], [53])**.** *Let $(G, [b], \{\mu\})$ be a local Shtuka datum with $G/\mathbf{Q}_p$ quasi-split, $b \in G(\breve{\mathbf{Q}}_p)$ basic and s-decent. Let $\ell \neq p$ be sufficiently generic*[9] *with respect to G.*

There are isomorphisms of $\mathscr{G}_{E_s} \times J(\mathbf{Q}_p)$-modules

$$H^*_{\text{ét},c}(\mathscr{F}^{\mathrm{wa}}_C, \mathbf{Z}/\ell^n) \simeq \bigoplus_{[w] \in W^\mu/\mathscr{G}_{E_s}} v^J_{P_{[w]}}(\mathbf{Z}/\ell^n) \otimes \rho_{[w]}(\mathbf{Z}/\ell^n)[-n_{[w]}],$$

$$H^*_{\text{ét,c,Hu}}(\mathscr{F}^{\mathrm{wa}}_C, \mathbf{Z}_\ell) \simeq \bigoplus_{[w] \in W^\mu/\mathscr{G}_{E_s}} v^J_{P_{[w]}}(\mathbf{Z}_\ell) \otimes \rho_{[w]}(\mathbf{Z}_\ell)[-n_{[w]}],$$

where $n_{[w]} = 2l_{[w]} + |\Delta \setminus I_{[w]}|$ and $H^*_{\text{ét,c,Hu}}$ denotes Huber's compactly supported cohomology. In particular, the action of $J(\mathbf{Q}_p)$ on $H^*_{\text{ét,c,Hu}}(\mathscr{F}^{\mathrm{wa}}_C, \mathbf{Z}_\ell)$ is smooth.

Our main result is the following computation.

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[9] See [51, Sec. 1] for the definition.

Theorem 1.2. *Let $(G,[b],\{\mu\})$ be a local Shtuka datum with $G/\mathbf{Q}_p$ quasi-split, $b\in G(\breve{\mathbf{Q}}_p)$ basic and s-decent. Assume that $p\geq 5$.*

There are isomorphisms of $\mathscr{G}_{E_s}\times J(\mathbf{Q}_p)$-modules

$$H^*_{\text{ét,c}}(\mathscr{F}^{\text{wa}}_C,\mathbf{Z}/p^n)\simeq\bigoplus_{[w]\in W^\mu/\mathscr{G}_{E_s}} v^J_{P_{[w]}}(\mathbf{Z}/p^n)\otimes\rho_{[w]}(\mathbf{Z}/p^n)[-n_{[w]}],$$

$$H^*_{\text{ét,c}}(\mathscr{F}^{\text{wa}}_C,\mathbf{Z}_p)\simeq\bigoplus_{[w]\in W^\mu/\mathscr{G}_{E_s}} v^{J,\text{cont}}_{P_{[w]}}(\mathbf{Z}_p)\otimes\rho_{[w]}(\mathbf{Z}_p)[-n_{[w]}],$$

where $H^*_{\text{ét,c}}$ denotes the continuous compactly supported cohomology, $v^{J,\text{cont}}_{P_I}(\mathbf{Z}_p)=\varprojlim_n v^J_{P_I}(\mathbf{Z}/p^n)$ denotes continuous *Steinberg representations, and $\rho_{[w]}(\mathbf{Z}_p)=\varprojlim_n\rho_{[w]}(\mathbf{Z}/p^n)$.*

Remark 1.3. The result for torsion coefficients in Theorem 1.2 is analogous to the one of Orlik quoted above. The analog of Orlik's second isomorphism is false: in the case of the adic affine space $\mathbb{A}^1_{\mathbf{Q}_p}$, which is a period domain for the group $G=\mathbb{G}_{m,\mathbf{Q}_p}\times\mathbb{G}_{m,\mathbf{Q}_p}$, we obtain (in the appendix) the isomorphism

$$H^2_{\text{ét},c,\text{Hu}}(\mathbb{A}^1_C,\mathbf{Z}_p(1))\simeq(\mathscr{O}_{\mathbb{P}^1_C,\infty}/C)\oplus\mathbf{Z}_p\,,$$

where $\mathscr{O}_{\mathbb{P}^1_C,\infty}$ is the stalk of analytic functions at ∞. This result is to be compared with the isomorphism $H^2_{\text{ét},c,\text{Hu}}(\mathbb{A}^1_C,\mathbf{Z}_\ell(1))\simeq\mathbf{Z}_\ell$, for $\ell\neq p$. Note, moreover, that the action of $G(\mathbf{Q}_p)$ on $H^2_{\text{ét},c,\text{Hu}}(\mathbb{A}^1_C,\mathbf{Z}_p(1))$ is not smooth.

Remark 1.4. The case of $\mathbb{A}^1_{\mathbf{Q}_p}$ suggests that Huber's definition is not the right one for p-adic coefficients. On the other hand, the continuous compactly supported cohomology[10]

$$\mathrm{R}\Gamma_{\text{ét},c}(X,\mathbf{Z}_p):=\mathrm{R}\varprojlim_n\mathrm{R}\Gamma_{\text{ét},c}(X,\mathbf{Z}/p^n),$$

gives sensible results, as Theorem 1.2 shows. In this particular case, we have an isomorphism

$$H^i_{\text{ét,c}}(\mathscr{F}^{\text{wa}}_C,\mathbf{Z}_p)\simeq H^i_{\text{ét},c,\text{naive}}(\mathscr{F}^{\text{wa}}_C,\mathbf{Z}_p):=\varprojlim_n H^i_{\text{ét,c}}(\mathscr{F}^{\text{wa}}_C,\mathbf{Z}/p^n),\quad i\geq 0,$$

with the naive version of compactly supported cohomology. We note that in a recent preprint [45], Lan-Liu-Zhu prove a Poincaré duality for p-adic étale cohomology of adic spaces admitting a nice compactification and the compactly supported cohomology that they use is the naive one, which is equal to the continuous one in their setting because their torsion cohomology groups are finite (hence satisfy the Mittag-Leffler condition).

Note that one could use also the continuous compactly supported cohomology in the ℓ-adic case, $\ell\neq p$, instead of Huber's version. One would get continuous generalized Steinberg representations instead of smooth ones in Theorem 1.1, which would fit better with the objects appearing in the p-adic Langlands program such as Emerton's completed cohomology. That would also make a (topological) Poincaré duality possible for the spaces that we consider.

Remark 1.5. Moreover:

1. We expect that the hypothesis $p\geq 5$ in Theorem 1.2 is not needed. This hypothesis is made so that we can use Theorem 1.8 below, which almost surely holds for any p.
2. Let $p\geq 5$. Let $\mathbb{H}^d_{\mathbf{Q}_p}$ be the Drinfeld symmetric space of dimension d over $\mathbf{Q}_p$. Recall that $\mathbb{H}^d_{\mathbf{Q}_p}=\mathbb{P}^d_{\mathbf{Q}_p}\setminus\cup_{H\in\mathscr{H}}H$, where $\mathscr{H}$ is the set of $\mathbf{Q}_p$-rational hyperplanes. Set $G:=\mathbb{GL}_{d+1,\mathbf{Q}_p}$. Theorem 1.2 yields an isomorphism of $\mathscr{G}_{\mathbf{Q}_p}\times G(\mathbf{Q}_p)$-modules

$$H^i_{\text{ét},c}(\mathbb{H}^d_C,\mathbf{Z}/p^n)\simeq\mathrm{Sp}_{2d-i}(\mathbf{Z}/p^n)(d-i),$$

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[10] Instead of requiring a proper support for a compatible sequence of global sections we just take sequences of properly supported global sections.

where the generalized Steinberg representations $\mathrm{Sp}_j(\mathbf{Z}/p^n)$ are as defined in Section 6.1.3. Comparing this isomorphism with that of [13]:

$$H^i_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Z}/p^n) \simeq \mathrm{Sp}_i(\mathbf{Z}/p^n)^*(-i)$$

one finds an abstract duality of $\mathrm{GL}_{d+1}(\mathbf{Q}_p) \times \mathscr{G}_{\mathbf{Q}_p}$-representations:

$$H^i_{\text{ét},c}(\mathbb{H}^d_C, \mathbf{Z}/p^n)(d) \simeq H^{2d-i}_{\text{ét}}(\mathbb{H}^d_C, \mathbf{Z}/p^n)^*.$$

It seems likely that this abstract duality is induced by the cup-product with values in $H^{2d}_{\text{ét},c}(\mathbb{H}^d_C, \mathbf{Z}/p^n(d)) \simeq \mathbf{Z}/p^n$ but we did not verify this.

This suggests that Poincaré duality holds for $\mathscr{F}^{\mathrm{wa}}_C$ and that one can deduce from Theorem 1.2 a description of the étale cohomology $H^*_{\text{ét}}(\mathscr{F}^{\mathrm{wa}}_C, \mathbf{Z}/p^n)$ as $J(\mathbf{Q}_p) \times \mathscr{G}_{\mathbf{Q}_p}$-modules.

(3) Suppose moreover that μ is minuscule. Thanks to the work of Fargues-Fontaine [24], Kedlaya-Liu [38], and Scholze [63], we can define the *admissible locus* $\mathscr{F}^a \subset \mathscr{F}^{\mathrm{wa}}$, a partially proper open subset of $\mathscr{F}$ having the same classical points as $\mathscr{F}^{\mathrm{wa}}$, and a p-adic local system over it interpolating the Galois representations associated to these classical points by the theorem of Colmez-Fontaine. In some remarkable situations (which can be completely classified thanks to the work of Chen-Fargues-Shen [10] and Goertz-He-Nie [27]) we have $\mathscr{F}^{\mathrm{wa}} = \mathscr{F}^a$ and so the above theorem describes the p-adic étale cohomology with compact support of the admissible locus. For instance, this is the case for the quasi-split group $G = \mathrm{SO}(V, q)$, where $V = \mathbf{Q}_p^n$ endowed with the quadratic form $q(x_1, \dots, x_n) = x_1x_n + x_2x_{n-1} + \cdots + x_nx_1$, the minuscule cocharacter $\mu(z) = \mathrm{diag}(z, 1, \dots, 1, z^{-1})$, and the basic class $[b] = [1] \in B(G, \mu)$, for which $J = G$. The flag variety is then the quadric $\mathscr{F}$ over $\breve{\mathbf{Q}}_p$ with equation $q(x) = 0$ in projective space and we have $\mathscr{F}^{\mathrm{wa}} = \mathscr{F}^a = \mathscr{F} \setminus G(\mathbf{Q}_p)S$, where S is the Schubert variety with equations $x_{\lceil n/2\rceil+1} = \cdots = x_n = 0$ inside $\mathscr{F}$ (we learnt this example from Fargues). For $n = 21$, we obtain a very concrete description of the p-adic period domain for polarized K3 surfaces with supersingular reduction and the previous theorem yields its p-adic étale cohomology with compact support. In general, we do not know how to describe the ℓ-adic étale cohomology (with compact support) of $\mathscr{F}^a$, even for $\ell \neq p$.

1.3. **The proof of the main result.** We will sketch the proof of Theorem 1.2 in the torsion case; the continuous case follows by taking limits.

1.3.1. *The geometric part.* As we have already mentioned, the geometric part of the proof is analogous to Orlik's proof of the corresponding result with ℓ-torsion coefficients, for $\ell \neq p$. Our contribution here lies solely in the verification that all ℓ-torsion statements in Orlik's proof work in the p-torsion setting as well. That this was not guaranteed is shown by the fact that it fails in the ℓ-adic setting: Orlik's ℓ-adic proof for $\ell \neq p$ breaks down p-adically.

The argument goes as follows. One starts with the distinguished triangle (associated to the triple $(\mathscr{F}^{\mathrm{wa}}, \mathscr{F}, \partial\mathscr{F}^{\mathrm{wa}})$, $\partial\mathscr{F}^{\mathrm{wa}} := \mathscr{F} \setminus \mathscr{F}^{\mathrm{wa}}$)

$$\mathrm{R}\Gamma_{\text{ét},c}(\mathscr{F}^{\mathrm{wa}}_C, \mathbf{Z}/p^n) \longrightarrow \mathrm{R}\Gamma_{\text{ét}}(\mathscr{F}_C, \mathbf{Z}/p^n) \longrightarrow \mathrm{R}\Gamma_{\text{ét}}(\partial\mathscr{F}^{\mathrm{wa}}_C, \mathbf{Z}/p^n).$$

This reduces the computation of $H^*_{\text{ét},c}(\mathscr{F}^{\mathrm{wa}}_C, \mathbf{Z}/p^n)$ to that of $H^*_{\text{ét}}(\partial\mathscr{F}^{\mathrm{wa}}_C, \mathbf{Z}/p^n)$: one needs to prove an isomorphism (we omit the coefficients $\mathbf{Z}/p^n$ in the formula):

$$H^*_{\text{ét}}(\partial\mathscr{F}^{\mathrm{wa}}_C) \simeq \begin{cases} \bigoplus_{|\Delta\setminus I_{[w]}|=1} \big(i^J_{P_{[w]}} \otimes \rho_{[w]}[-2l_{[w]}]\big) \\ \bigoplus \\ \bigoplus_{|\Delta\setminus I_{[w]}|>1} \big(\rho_{[w]}[-2l_{[w]}] \oplus \big(v^J_{P_{[w]}} \otimes \rho_{[w]}[-2l_{[w]} - |\Delta \setminus I_{[w]}| + 1]\big)\big) \end{cases} \tag{1.6}$$

To do it, one stratifies the complement $\partial\mathscr{F}^{\mathrm{wa}}$ by Schubert varieties whose cohomology is easy to compute. More precisely, one uses the Faltings and Totaro description of weak admissibility as a semistability condition: the period domain $\mathscr{F}^{\mathrm{wa}}$ is the locus of semistability in $\mathscr{F}$ and the complement $\partial\mathscr{F}^{\mathrm{wa}}$ is the locus in $\mathscr{F}$, where semistability fails. To test semistability one applies the

Hilbert-Mumford criterion: for a field extension $K/\breve{E}$, $x \in \mathscr{F}(K)$ is semistable (hence $x \in \mathscr{F}^{\rm wa}(K)$) if and only if $\mu(x,\lambda) \geq 0$, for all $\lambda \in X_*(J)^{\mathscr{G}_F}$. Here $\mu(-,-)$ is the slope function associated to a linearization of the action of J.

The slope function, a priori convex on each chamber of the spherical building $\mathscr{B}(J_{\rm der})$, is actually affine. This implies that, in the Hilbert-Mumford criterion, it is enough to test the 1-parameter subgroups associated to the relative simple roots and their conjugates. This leads to the stratification

$$\partial \mathscr{F}^{\rm wa} = Z_1 \supset \cdots \supset Z_{i-1} \supset Z_i \supset Z_{i+1} \supset \cdots$$

that is defined in the following way. For $\lambda \in X_*(J)_{\mathbf{Q}}$, let Y_λ be the locus in $\mathscr{F}$, where λ damages the semistability condition. For $I \subset \Delta$, let $Y_I := \cap_{\alpha \notin I} Y_{\omega_\alpha}$ be the associated Schubert variety. Then the locus Z_i of $\partial\mathscr{F}^{\rm wa}$, where the semistability fails to the degree at least i, can be described as

$$Z_i = \bigcup_{|\Delta \setminus I| = i} Z_I, \quad Z_I := J(\mathbf{Q}_p)\cdot Y_I^{\rm ad}.$$

We note that Z_I is a closed pseudo-adic subspace of $\partial\mathscr{F}^{\rm wa}$. In particular, so is $\partial\mathscr{F}^{\rm wa} = Z_1 = \bigcup_{|\Delta\setminus I|=1} Z_I$.

Having this stratification, by a procedure akin to a closed Mayer-Vietoris, one obtains an *acyclic* complex of sheaves on $\partial\mathscr{F}^{\rm wa}_C$, called *the fundamental complex*,

$$0 \to \mathbf{Z}/p^n \to \bigoplus_{|\Delta\setminus I|=1} (\mathbf{Z}/p^n)_I \to \bigoplus_{|\Delta\setminus I|=2} (\mathbf{Z}/p^n)_I \to \cdots \to \bigoplus_{|\Delta\setminus I|=|\Delta|-1} (\mathbf{Z}/p^n)_I \to (\mathbf{Z}/p^n)_\varnothing \to 0,$$

where $(\mathbf{Z}/p^n)_I$ denotes the constant sheaf $\mathbf{Z}/p^n$ evaluated[11] on $Z_{I,C}$. This complex yields a spectral sequence

$$E_1^{i,j} = \bigoplus_{|\Delta\setminus I|=i+1} H^i_{\rm \acute{e}t}(\partial\mathscr{F}^{\rm wa}_C, (\mathbf{Z}/p^n)_I) \Rightarrow H^{i+j}_{\rm \acute{e}t}(\partial\mathscr{F}^{\rm wa}_C, \mathbf{Z}/p^n). \tag{1.7}$$

Using the fact that $P_I(\mathbf{Q}_p)$ is the stabilizer of Y_I in $J(\mathbf{Q}_p)$ and $X_I = J(\mathbf{Q}_p)/P_I(\mathbf{Q}_p)$ one computes that

$$\begin{aligned} H^i_{\rm \acute{e}t}(\partial\mathscr{F}^{\rm wa}_C, (\mathbf{Z}/p^n)_I) &\simeq {\rm LC}(X_I, H^i_{\rm \acute{e}t}(Y_{I,C}, \mathbf{Z}/p^n)) \simeq i^J_{P_I}(\mathbf{Z}/p^n) \otimes H^i_{\rm \acute{e}t}(Y_{I,C}, \mathbf{Z}/p^n) \\ &\simeq i^J_{P_I}(\mathbf{Z}/p^n) \otimes \Big(\bigoplus_{[w]\in\Omega_I} \rho_{[w]}(\mathbf{Z}/p^n)[-2l_{[w]}]\Big). \end{aligned}$$

Here Ω_I is a subset of $W^\mu/\mathscr{G}_{E_s}$ (see Section 5.5). The third isomorphism is obtained by the classical computation of the cohomology of Schubert varieties. Via a simple Galois-theoretic weight argument, this computation implies that the above spectral sequence degenerates at E_2. Using results of Grosse-Klönne [28], Herzig [35], and Ly [46] on generalized Steinberg representations mod p, one can also compute the E_2 terms: they are equal to the terms on the right hand side of the formula (1.6).

1.3.2. *The group theoretic part.* It follows from the above section that the grading of $H^*_{\rm \acute{e}t}(\partial\mathscr{F}^{\rm wa}_C, \mathbf{Z}/p^n)$ associated to the filtration induced by the spectral sequence (1.7) is isomorphic to the right hand side of (1.6). It remains to show that this filtration splits. And this is where things get much harder for p-torsion coefficients than for the ℓ-torsion ones. Splitting this filtration essentially comes down to understanding Ext groups between generalized Steinberg representations with p-torsion coefficients. Fortunately, it suffices to deal with ${\rm Ext}^1$'s, which are the only ones we can handle, contrary to the usual theory with complex coefficients (adapted to the ℓ-adic setting by Orlik [52] and Dat [16]). It is indeed a well-known phenomenon in the theory of smooth mod p representations of p-adic reductive groups that Ext groups can be very hard to compute, since most of the techniques for complex or ℓ-adic coefficients fail.

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[11] We simplify for the sake of the introduction; see Section 6.2.1 for details.

Before stating the key result that allows us to split the filtration, let us briefly explain the argument for complex or ℓ-torsion coefficients and point out the difficulties occurring for p-torsion coefficients. Let R be one of the rings $\mathbf{C}, \mathbf{Z}/\ell^n, \mathbf{Z}/p^n$ (ℓ being sufficiently generic with respect to G). One can construct an acyclic complex

$$0 \to i^J_\Delta(R) \to \bigoplus_{\substack{I\subset I'\subset\Delta\\ |\Delta\setminus I'|=1}} i^J_{P_{I'}}(R) \to \cdots \to \bigoplus_{\substack{I\subset I'\subset\Delta\\ |I'\setminus I|=1}} i^J_{P_{I'}} \to i^J_{P_I}(R) \to v^J_{P_I}(R) \to 0.$$

For $R = \mathbf{C}$ or $\mathbf{Z}/\ell^n$ this is a rather standard result, and it also works for $\mathbf{Z}/p^n$ thanks to the above-mentioned work of Grosse-Klönne, Herzig, and Ly (the acyclicity of this complex is also crucial in computing the E_2 terms of the above spectral sequence). Suppose that $R \neq \mathbf{Z}/p^n$. A spectral sequence argument reduces the computation of Ext groups between $v^J_{P_{I_1}}(R)$ and $v^J_{P_{I_2}}(R)$ to the computation of $\mathrm{Ext}^*_J(i^J_{P_I}(R), i^J_{P_{I'}}(R))$. The exactness of the Jacquet functor (which fails when $R = \mathbf{Z}/p^n$) reduces the problem to understanding extensions between the Jacquet module of $i^J_{P_I}(R)$ (which can be understood by the Bernstein-Zelevinsky geometric lemma) and the trivial representation. After several other relatively standard but technical arguments one reduces everything to the computation of $H^*(J(\mathbf{Q}_p), i^J_{P_I}(R)) = H^*(M_I(\mathbf{Q}_p), R)$, where M_I is the Levi quotient of the standard parabolic P_I. Thus we are reduced to computing $H^*(G(\mathbf{Q}_p), R)$ for a reductive group G over $\mathbf{Q}_p$, which can be done using the contractibility of the Bruhat-Tits building of G *and* the fact that $i^G_K(1)$ is injective as smooth representation whenever K is a compact open subgroup of G (since passage to K-invariants is exact). This again fails when $R = \mathbf{Z}/p^n$. Actually, it is an interesting problem to compute $H^*(G(\mathbf{Q}_p), \mathbf{Z}/p^n)$ for a reductive group G over $\mathbf{Q}_p$. Unfortunately we don't have much to say about this except to mention that the computation of $H^*(\mathbb{GL}_n(\mathbf{Z}_p), \mathbf{Z}/p)$ seems rather complicated: Lazard's theory allows one to compute $H^*(1 + pM_n(\mathbf{Z}_p), \mathbf{Z}/p)$ (at least when $1+pM_n(\mathbf{Z}_p)$ is a uniform pro-p group), so one is reduced to the computation of $H^*(\mathbb{GL}_n(\mathbf{Z}/p), \mathbf{Z}/p)$, a well-known open problem.

The previous paragraph makes it clear that a new idea is needed in order to compute extensions between generalized Steinberg representations modulo p. More generally, let F be a local field of residue characteristic p, let G be the group of F-points of a connected reductive algebraic F-group, and let R be an Artinian commutative ring in which p is nilpotent. We will only focus on the computation of Ext^1, which is enough for our needs. The computation of higher Ext groups seems to require new ideas. All Ext groups below are computed in the category of smooth representations of G with coefficients in R.

The generalized Steinberg representations are parametrized by the subsets of a basis Δ of the relative root system of G. For any subset $I \subset \Delta$, we let P_I be the group of F-points of the corresponding standard parabolic subgroup of G and set $i^G_{P_I} = \mathrm{LC}(P_I\backslash G, R)$. If $J \supset I$ is another subset, then there is an injection $i^G_{P_J} \hookrightarrow i^G_{P_I}$ which is induced by the natural surjection $P_I\backslash G \twoheadrightarrow P_J\backslash G$. The generalized Steinberg representation with respect to I is the quotient

$$v^G_{P_I} = i^G_{P_I} / \sum_{J\supsetneq I} i^G_{P_J}.$$

In the case $I = \varnothing$ we obtain the ordinary Steinberg representation denoted St. In the case $I = \Delta$ we obtain the trivial representation denoted 1. The key result needed to split the abutment filtration of the spectral sequence (1.7) is then:

Theorem 1.8. *Assume that $p \geq 5$. Let $I, J \subset \Delta$. If $|(I \cup J) \setminus (I \cap J)| \geq 2$, then $\mathrm{Ext}^1_G(v^G_{P_I}, v^G_{P_J}) = 0$.*

Remark 1.9. (1) We expect Theorem 1.8 to hold true for all p. Actually, we prove the result in almost all cases when $p = 3$ and in some cases when $p = 2$. From our computations, we see that the Ext^1 group is always killed by 3 when $p = 3$ and by 16 when $p = 2$ (see Remarks 2.7 and 2.40). In particular, if $\mathrm{char}(F) = 0$ and E/F is a finite extension, then the analogous Ext^1's in the category of admissible unitary continuous G-representations on E-Banach spaces vanishes for all p (see [31, Prop. 5.3.1] and [30, Lemme 3.3.3]).

(2) We refer the reader to Section 2.2 for an overview of the rather technical proof of the theorem. The most difficult part is the computation of $H^1(G, \mathrm{St})$ and one can actually reduce the theorem to this computation by rather painful dévissage arguments involving Emerton's ordinary parts functor and its derived functors.

If $I = J$, then the R-module $\mathrm{Ext}^1_G(v^G_{P_I}, v^G_{P_J})$ has been computed in [34] without any assumption (see Proposition 8 in loc. cit.). In the course of the proof of Theorem 1.8, we also treat the case $J = I \sqcup \{\alpha\}$ under a very mild assumption (which is always satisfied if $p \geq 5$) and we reduce the remaining case $I = J \sqcup \{\alpha\}$ to the special case where $\Delta = \{\alpha\}$ when $p \neq 2$. We treat the latter case under some assumption on G. In particular, we obtain the following result.

Theorem 1.10. *Assume that $G = \mathrm{GL}_n(D)$ for some division algebra D over F and let $I, J \subset \Delta$. If $J \not\subset I$ assume that $D \neq \mathbf{Q}_2$.*

1. *If $J = I \sqcup \{\alpha\}$, then the R-module $\mathrm{Ext}^1_G(v^G_{P_I}, v^G_{P_J})$ is free of rank 1.*
2. *If $I = J \sqcup \{\alpha\}$, then there is an R-linear isomorphism*

$$\mathrm{Ext}^1_G(v^G_{P_I}, v^G_{P_J}) \simeq \mathrm{Hom}(E^*, R),$$

where E denotes the center of D.

3. *If $|(I \cup J) \setminus (I \cap J)| \geq 2$, then $\mathrm{Ext}^1_G(v^G_{P_I}, v^G_{P_J}) = 0$.*

Remark 1.11. (1) In contrast, we do not know how to compute the R-module $\mathrm{Ext}^1_G(\mathrm{St}, 1)$ when $G = \mathbb{GL}_2(\mathbf{Q}_2)$, or $\mathrm{Ext}^1_G(v^G_{P_1}, v^G_{P_2})$ when $G = \mathbb{GL}_3(\mathbf{Q}_2)$ and P_1, P_2 denote the two maximal proper standard parabolic subgroups of G.

(2) A locally analytic version of part (2) of Theorem 1.10 (for the group $\mathbb{GL}_n$) is established in the work of Ding [18] and generalized to split reductive groups by Gehrmann [26]. Higher Ext groups are computed by Orlik and Strauch [55], for split reductive groups and in a suitable category of locally analytic representations (but not in the category of admissible locally analytic representations). We note that a vanishing result for Ext^1 in the locally analytic world gives a corresponding vanishing result in the context of admissible Banach representations, since the continuous generalized Steinberg representations are the universal unitary completions of their locally analytic vectors. We thank Lennart Gehrmann for pointing out the references above. Let us mention though that the vanishing mod p in Theorem 1.8 is crucial for the proof of Theorem 1.2: the corresponding result for Banach representations is not sufficient for our needs.

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2. Extensions between generalized Steinberg representations

Let F be a local field of residue characteristic p, G be the group of F-points of a connected reductive algebraic F-group, and R be an artinian commutative ring in which p is nilpotent. In this section we prove a vanishing result for the Ext^1 groups between generalized Steinberg representations in the category of smooth G-representations with coefficients in R (compare [52]). More precisely, we will prove Theorems 1.8 and 1.10 from the introduction.

2.1. Notation.

Let us fix some notation for this section. We fix a separable closure $\overline{F}$ of F and let $\mathscr{G}_F = \mathrm{Gal}(\overline{F}/F)$. Let $\varepsilon : \mathbf{Q}_p^* \to \mathbf{Z}_p^*$ denote the p-adic cyclotomic character and $\bar{\varepsilon} : \mathbf{Q}_p^* \to \mathbf{F}_p^*$ denote its reduction mod p.

A linear algebraic F-group will be written with a boldface letter like $\boldsymbol{H}$ and its group of F-points will be denoted by the corresponding ordinary letter $H = \boldsymbol{H}(F)$. We will write $\boldsymbol{Z_H}$ for the center of $\boldsymbol{H}$.

Let $\boldsymbol{G}$ be a connected reductive algebraic F-group. Fix a maximal split torus $\boldsymbol{S} \subset \boldsymbol{G}$ and a minimal parabolic subgroup $\boldsymbol{B} \subset \boldsymbol{G}$ containing $\boldsymbol{S}$. Let $\boldsymbol{\mathscr{Z}}$ be the centralizer of $\boldsymbol{S}$ in $\boldsymbol{G}$, which is a Levi subgroup of $\boldsymbol{B}$, and $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{B}$. Let

$$\Phi \supset \Phi^+ \supset \Delta$$

denote the corresponding subsets of relative roots, positive roots, simple roots in $X^*(\boldsymbol{S})$. Any $\alpha \in \Delta$ whose corresponding root subgroup $\boldsymbol{U}_\alpha$ is one-dimensional extends to a character of $\boldsymbol{\mathscr{Z}}$ which will be denoted $\widetilde{\alpha}$. Let $\boldsymbol{\mathscr{N}}$ be the normalizer of $\boldsymbol{S}$ in $\boldsymbol{G}$ and

$$W = \boldsymbol{\mathscr{N}}/\boldsymbol{\mathscr{Z}} = \mathscr{N}/\mathscr{Z}$$

be the relative Weyl group of $\boldsymbol{G}$. We let $s_\alpha \in W$ denote the simple reflection associated to $\alpha \in \Delta$ and we write $l_w \in \mathbf{N}$ for the length with respect to Δ of an element $w \in W$. For $w \in W$, we let

$$\overline{\boldsymbol{U}}_w = \overline{\boldsymbol{U}} \cap w^{-1}\boldsymbol{U}w \quad \text{and} \quad \overline{\boldsymbol{B}}_w = \boldsymbol{\mathscr{Z}}\overline{\boldsymbol{U}}_w.$$

For $I \subset \Delta$, we let $\boldsymbol{P_I} \subset \boldsymbol{G}$ be the corresponding parabolic subgroup containing $\boldsymbol{B}$, $\boldsymbol{M_I}$ be the Levi factor of $\boldsymbol{P_I}$ containing $\boldsymbol{S}$, $\boldsymbol{N_I}$ be the unipotent radical of $\boldsymbol{P_I}$, $\boldsymbol{Z_I}$ be the center of $\boldsymbol{M_I}$, and we set $\boldsymbol{B}_I = \boldsymbol{M_I} \cap \boldsymbol{B}$ (a minimal parabolic subgroup of $\boldsymbol{M_I}$ containing $\boldsymbol{S}$) and $\boldsymbol{U}_I = \boldsymbol{M_I} \cap \boldsymbol{U}$ (the unipotent radical of $\boldsymbol{B_I}$). We let $\overline{\boldsymbol{P}}_I = \boldsymbol{M_I}\overline{\boldsymbol{N}}_I$ be the parabolic subgroup of $\boldsymbol{G}$ opposite to $\boldsymbol{P_I}$ with respect to $\boldsymbol{M_I}$, and we set $\overline{\boldsymbol{B}}_I = \boldsymbol{M_I} \cap \overline{\boldsymbol{B}}$ and $\overline{\boldsymbol{U}}_I = \boldsymbol{M_I} \cap \overline{\boldsymbol{U}}$. We let $\boldsymbol{\mathscr{N}}_I = \boldsymbol{M_I} \cap \boldsymbol{\mathscr{N}}$ be the normalizer of $\boldsymbol{S}$ in $\boldsymbol{M_I}$ and

$$W_I = \boldsymbol{\mathscr{N}}_I/\boldsymbol{\mathscr{Z}} = \mathscr{N}_I/\mathscr{Z}$$

be the relative Weyl group of $\boldsymbol{M_I}$. We let $\widetilde{W_I} \subset W$ be the set of representatives of minimal length of the right cosets $W_I \backslash W$, $w_{I,0} \in W_I$ be the longest element, and we set

$$\widehat{W_I} = w_{I,0}\widetilde{W_I} \setminus \bigcup_{J \supsetneq I} w_{J,0}\widetilde{W_J}.$$

For $w_I \in W_I$, we set

$$\overline{\boldsymbol{U}}_{I,w_I} = \overline{\boldsymbol{U}}_I \cap w_I^{-1}\boldsymbol{U}_I w_I \quad \text{and} \quad \overline{\boldsymbol{B}}_{I,w_I} = \boldsymbol{\mathscr{Z}}\overline{\boldsymbol{U}}_{I,w_I}$$

When $I = \{\alpha\}$ (resp. $I = \Delta \setminus \{\alpha\}$), we rather write $\boldsymbol{P}_\alpha$, $\boldsymbol{M}_\alpha$, $\boldsymbol{N}_\alpha$, etc. (resp. $\boldsymbol{P}^\alpha$, $\boldsymbol{M}^\alpha$, $\boldsymbol{N}^\alpha$, etc.).

All representations will be smooth and have coefficients in R, and all maps between R-modules will be R-linear. Given a locally profinite space X, we let $\mathrm{LC}(X) = \mathrm{LC}(X, R)$ be the R-module of locally constant functions on X with coefficients in R and we let $\mathrm{supp}(f) = f^{-1}(\{0\})$ denote the (open and closed) support of an element $f \in \mathrm{LC}(X)$. Given a closed subgroup $\overline{\boldsymbol{U}}'$ of $\overline{\boldsymbol{U}}$ stable under conjugation by $\boldsymbol{\mathscr{Z}}$, we define the R-submodules

$$\mathrm{LC}_c(\overline{U}') \subset \mathrm{LC}_{\mathrm{u}}(\overline{U}') \subset \mathrm{LC}(\overline{U}')$$

of compactly supported locally constant functions and uniformly locally constant functions. These are endowed with the smooth action of $\overline{U}'$ by right translation and the smooth action of $\mathscr{Z}$ defined by $z \cdot f : \bar{u}' \mapsto f(z^{-1}\bar{u}'z)$. Given a closed subgroup H of G and a smooth H-representation σ, we define smooth G-representations

$$\mathrm{Ind}_H^G(\sigma) = \{f : G \to \sigma \mid \exists K_f \subset G \text{ open subgroup s.t. } f(hgk) = h \cdot f(g)\ \forall h \in H, g \in G, k \in K_f\},$$
$$\text{c-}\mathrm{Ind}_H^G(\sigma) = \{f \in \mathrm{Ind}_H^G(\sigma) \mid \mathrm{supp}(f) \text{ is compact modulo } H\}.$$

Let 1 be the trivial representation of any locally profinite group. For any subset $I \subset \Delta$, let

$$i_{P_I}^G = \mathrm{LC}(P_I \backslash G) = \mathrm{Ind}_{P_I}^G(1) = \text{c-}\mathrm{Ind}_{P_I}^G(1).$$

If $J \supset I$ is another subset, then there is an injection $i_{P_J}^G \hookrightarrow i_{P_I}^G$ which is induced by the natural surjection $P_I \backslash G \twoheadrightarrow P_J \backslash G$. The generalized Steinberg representation with respect to I is the quotient

$$v_{P_I}^G = i_{P_I}^G / \sum_{J \supsetneq I} i_{P_J}^G.$$

In the case $I = \varnothing$ we obtain the ordinary Steinberg representation denoted St. In the case $I = \Delta$ we obtain the trivial representation 1.

2.2. **The general strategy.** We fix two subsets $I, J \subset \Delta$. First, recall the computation of the R-module $\mathrm{Hom}_G(v_{P_I}^G, v_{P_J}^G)$.

Proposition 2.1 (Grosse-Klönne, Herzig, Ly)**.** *There is an R-linear isomorphism*

$$\mathrm{Hom}_G(v_{P_I}^G, v_{P_J}^G) \simeq \begin{cases} R & \textit{if } I = J, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $I = J$, then the result is a special case of [34, Cor. 5]. If $I \neq J$, then by dévissage the result reduces to the case where R is a field of characteristic p, which is proved by Grosse-Klönne [28] and Herzig [35] when $\boldsymbol{G}$ is split, and by Ly [46] in general. □

Now, using the results of the next paragraphs, we compute the R-module $\mathrm{Ext}_G^1(v_{P_I}^G, v_{P_J}^G)$ when $I \neq J$. We treat the two following cases separately.

Case $J \not\subset I$. Let $\alpha \in (\Delta \setminus I) \cap J$. If $F = \mathbf{Q}_p$ and $\dim \boldsymbol{U}_\alpha = 1$ assume that $\bar{\varepsilon} \circ \widetilde{\alpha} \neq 1$ (this is always true if $p \geq 5$, and actually we only need this assumption when $J \setminus \{\alpha\} = I \cap \{\alpha\}^\perp$, see Remark 2.7). There is an R-linear isomorphism (see (2.3)):

$$\mathrm{Ext}_G^1(v_{P_I}^G, v_{P_J}^G) \simeq \begin{cases} R & \text{if } J = I \sqcup \{\alpha\}, \\ 0 & \text{otherwise.} \end{cases}$$

In the case $J = I \sqcup \{\alpha\}$, the R-module $\mathrm{Ext}_G^1(v_{P_I}^G, v_{P_J}^G)$ is generated by the class of $\mathrm{Ind}_{P^\alpha}^G(v_{M^\alpha \cap P_I}^{M^\alpha})$ (see (2.2)).

Case $J \subsetneq I$. Assume that $p \neq 2$. We prove the following results.

1. If $I = J \sqcup \{\alpha\}$ and the adjoint action of $\mathscr{Z}$ on $\overline{U}_\alpha \setminus \{1\}$ is transitive (e.g. if $G = \mathrm{GL}_n(D)$ for some division algebra D over F), then there is an R-linear isomorphism
$$\mathrm{Ext}_G^1(v_{P_I}^G, v_{P_J}^G) \xrightarrow{\sim} \mathrm{Ext}_{\mathscr{Z}}^1(1,1)^{s_\alpha = -1}.$$
2. If $|I \setminus J| > 1$, then $\mathrm{Ext}_G^1(v_{P_I}^G, v_{P_J}^G) = 0$.

We proceed by induction on $|\Delta \setminus I|$. If $I \neq \Delta$, then we fix $\alpha \in \Delta \setminus I$ and we can use the induction hypothesis twice with the exact sequence of R-modules (see (2.4))

$$0 \to \mathrm{Ext}_G^1(v_{P_I}^G, v_{P_J}^G) \to \mathrm{Ext}_{M^\alpha}^1(v_{M^\alpha \cap P_I}^{M^\alpha}, v_{M^\alpha \cap P_J}^{M^\alpha}) \to \mathrm{Ext}_G^1(v_{P_{I \sqcup \{\alpha\}}}^G, v_{P_J}^G).$$

In the case $I = \Delta$, we proceed by induction on $|J|$. If $J \neq \varnothing$, then we fix $\alpha \in J$ and we can use the induction hypothesis twice with the exact sequence of R-modules (see (2.8))

$$0 \to \mathrm{Ext}_G^1(1, v_{P_J}^G) \to \mathrm{Ext}_{M^\alpha}^1(1, v_{P_{J \setminus \{\alpha\}}}^{M^\alpha}) \to \mathrm{Ext}_G^1(1, v_{P_{J \setminus \{\alpha\}}}^G).$$

In the base case $I = \Delta$ and $J = \varnothing$, the results follow from Propositions 2.41 and 2.43 (this is the most difficult part of the argument).

This completes the proof of Theorem 1.8. We turn to Theorem 1.10. Assume that $G = \mathrm{GL}_n(D)$ for some division algebra D over F. In the case $J \not\subset I$, the above assumption becomes $D \neq \mathbf{Q}_2$. In the case $J \subsetneq I$, the above results hold true when $p = 2$ (see Remarks 2.42 and 2.44 for the base case).

2.3. **Reduction to the case $I = \Delta$ and $J = \varnothing$.** For $I \subset \Delta$, we let $\mathrm{Ind}_{P_I}^G$ denote the smooth parabolic induction functor and $\mathrm{Ord}_{\overline{P}_I}$ denote the ordinary part functor [19, 67]. Moreover, we let $H^*\mathrm{Ord}_{\overline{P}_I}$ denote the δ-functor of higher ordinary parts [20] when $\mathrm{char}(F) = 0$ and we simply set $H^1\mathrm{Ord}_{\overline{P}_I} = 0$ when $\mathrm{char}(F) = p$.

2.3.1. *Reduction to the case $I = \Delta$.* Assume that $I \neq \Delta$ and let $\alpha \in \Delta \setminus I$. By exactness and transitivity of parabolic induction, there is a short exact sequence of smooth G-representations

$$0 \to v_{P_{I \sqcup \{\alpha\}}}^G \to \mathrm{Ind}_{P^\alpha}^G(v_{M^\alpha \cap P_I}^{M^\alpha}) \to v_{P_I}^G \to 0 \tag{2.2}$$

which induces an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_G(v_{P_I}^G, v_{P_J}^G) \to \mathrm{Hom}_G(\mathrm{Ind}_{P^\alpha}^G(v_{M^\alpha \cap P_I}^{M^\alpha}), v_{P_J}^G) \to \mathrm{Hom}_G(v_{P_{I \sqcup \{\alpha\}}}^G, v_{P_J}^G) \\ &\to \mathrm{Ext}_G^1(v_{P_I}^G, v_{P_J}^G) \to \mathrm{Ext}_G^1(\mathrm{Ind}_{P^\alpha}^G(v_{M^\alpha \cap P_I}^{M^\alpha}), v_{P_J}^G) \to \mathrm{Ext}_G^1(v_{P_{I \sqcup \{\alpha\}}}^G, v_{P_J}^G). \end{aligned}$$

By [19, Th. 4.4.6] and [67, Cor. 8.3], there is an isomorphism

$$\mathrm{Hom}_G(\mathrm{Ind}_{P^\alpha}^G(v_{M^\alpha \cap P_I}^{M^\alpha}), v_{P_J}^G) \simeq \mathrm{Hom}_{M^\alpha}(v_{M^\alpha \cap P_I}^{M^\alpha}, \mathrm{Ord}_{\overline{P}^\alpha}(v_{P_J}^G))$$

and by [20, (3.7.6)] and [32, Cor. 2], there is a short exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}_{M^\alpha}^1(v_{M^\alpha \cap P_I}^{M^\alpha}, \mathrm{Ord}_{\overline{P}^\alpha}(v_{P_J}^G)) \to \mathrm{Ext}_G^1(\mathrm{Ind}_{P^\alpha}^G(v_{M^\alpha \cap P_I}^{M^\alpha}), v_{P_J}^G) \\ &\to \mathrm{Hom}_{M^\alpha}(v_{M^\alpha \cap P_I}^{M^\alpha}, H^1\mathrm{Ord}_{\overline{P}^\alpha}(v_{P_J}^G)). \end{aligned}$$

By [2, Th. 6.1(ii)], there is an M^α-equivariant isomorphism

$$\mathrm{Ord}_{\overline{P}^\alpha}(v_{P_J}^G) \simeq \begin{cases} v_{M^\alpha \cap P_J}^{M^\alpha} & \text{if } \alpha \notin J, \\ 0 & \text{otherwise.} \end{cases}$$

Using Lemma 2.5 below and taking into account Proposition 2.1, we obtain the following results.

Case $\alpha \in J$. If $F = \mathbf{Q}_p$ and $\dim \boldsymbol{U}_\alpha = 1$ assume that $\bar{\varepsilon} \circ \widetilde{\alpha} \neq 1$. There is an isomorphism

$$\mathrm{Ext}_G^1(v_{P_I}^G, v_{P_J}^G) \simeq \begin{cases} R & \text{if } J = I \sqcup \{\alpha\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

In the case $J = I \sqcup \{\alpha\}$, the R-module $\mathrm{Ext}_G^1(v_{P_I}^G, v_{P_J}^G)$ is generated by the class of (2.2).

Case $\alpha \notin J$. There is an exact sequence

$$0 \to \mathrm{Ext}_G^1(v_{P_I}^G, v_{P_J}^G) \to \mathrm{Ext}_{M^\alpha}^1(v_{M^\alpha \cap P_I}^{M^\alpha}, v_{M^\alpha \cap P_J}^{M^\alpha}) \to \mathrm{Ext}_G^1(v_{P_{I \sqcup \{\alpha\}}}^G, v_{P_J}^G). \tag{2.4}$$

Lemma 2.5. (1) *$H^1\mathrm{Ord}_{\overline{P}^\alpha}(v_{P_J}^G) \neq 0$ if and only if $F = \mathbf{Q}_p$, $\dim \boldsymbol{U}_\alpha = 1$, and $\alpha \in J$.*

(2) *Assume that $F = \mathbf{Q}_p$, $\dim \boldsymbol{U}_\alpha = 1$, and $\alpha \in J$. If $\bar{\varepsilon} \circ \widetilde{\alpha} \neq 1$, then*

$$\mathrm{Hom}_{M^\alpha}(v_{M^\alpha \cap P_I}^{M^\alpha}, H^1\mathrm{Ord}_{\overline{P}^\alpha}(v_{P_J}^G)) = 0.$$

Proof. First, we review the filtration on $v_{P_J}^G$ studied in [30] (in loc. cit. $\boldsymbol{G}$ is split, but the results extend verbatim to any $\boldsymbol{G}$; they can also be extracted from [2, §7.2]). The Bruhat decomposition

$$G = \bigsqcup_{w \in W} Bw\overline{B}$$

induces a filtration on i_B^G by $\overline{B}$-subrepresentations with graded pieces I_w indexed by W, and there is a $\overline{B}_w$-equivariant isomorphism $I_w \simeq \mathrm{LC_c}(\overline{U}_w)$ for all $w \in W$. Using the G-equivariant morphisms

$$i_B^G \hookleftarrow i_{P_J}^G \twoheadrightarrow v_{P_J}^G,$$

the filtration on i_B^G induces a filtration on $v_{P_J}^G$ by $\overline{B}$-subrepresentations whose graded pieces are the $\overline{B}$-representations $I_{\widehat{w}_J}$ for all $\widehat{w}_J \in \widehat{W}_J$ (see [30, Prop. 2.3.4]; see also [34, §5] using the $\overline{B}$-equivariant isomorphism

$$\text{c-Ind}_{P_J}^{P_J \widetilde{w}_J \overline{B}} 1 \simeq \text{c-Ind}_{B}^{B w_{J,0} \widetilde{w}_J \overline{B}} 1$$

for any $\widetilde{w}_J \in \widetilde{W}_J$, see [29, §2.3]). The filtration on $v_{P_J}^G$ induces a filtration on $H^1\mathrm{Ord}_{\overline{P}^\alpha}(v_{P_J}^G)$ by $\overline{B}^\alpha$-subrepresentations whose graded pieces are the $\overline{B}^\alpha$-representations $H^1\mathrm{Ord}_{\overline{P}^\alpha}(I_{\widehat{w}_J})$ for all $\widehat{w}_J \in \widehat{W}_J$ (see the proof of [30, Prop. 2.3.6]).

Now we give the partial computation of $H^1\mathrm{Ord}_{\overline{P}^\alpha}(I_{\widehat{w}_J})$ from [33] (in loc. cit. $\boldsymbol{G}$ is split, but the results extend verbatim to any $\boldsymbol{G}$; they can also be extracted from [31, §3.3]). If $F = \mathbf{Q}_p$ and $\dim \boldsymbol{U}_\alpha = 1$, then for any $\widehat{w}_J \in \widehat{W}_J$, using the unique decomposition $\widehat{w}_J = (\widetilde{w}^\alpha)^{-1} w^\alpha$ with $\widetilde{w}^\alpha \in \widetilde{W}^\alpha$ and $w^\alpha \in W^\alpha$, there is a $\overline{B}_{w^\alpha}^\alpha$-equivariant isomorphism

$$H^1\mathrm{Ord}_{\overline{P}^\alpha}(I_{\widehat{w}_J}) \simeq \begin{cases} \mathrm{LC_c}(\overline{U}_{w^\alpha}^\alpha) \otimes (\varepsilon \circ \widetilde{\alpha}) & \text{if } \widetilde{w}^\alpha = s_\alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

In particular, $H^1\mathrm{Ord}_{\overline{P}^\alpha}(v_{P_J}^G) \neq 0$ if and only if $(s_\alpha W^\alpha) \cap \widehat{W}_J \neq \varnothing$, and the latter condition is equivalent to $\alpha \in J$. If either $F \neq \mathbf{Q}_p$ or $\dim \boldsymbol{U}_\alpha > 1$, then $H^1\mathrm{Ord}_{\overline{P}^\alpha}(I_{\widehat{w}_J}) = 0$ for all $\widehat{w}_J \in \widehat{W}_J$ so that $H^1\mathrm{Ord}_{\overline{P}^\alpha}(v_{P_J}^G) = 0$. This proves (1).

We turn to (2). Assume that $F = \mathbf{Q}_p$, $\dim \boldsymbol{U}_\alpha = 1$, and $\alpha \in J$. By definition of $v_{M^\alpha \cap P_I}^{M^\alpha}$, there is an injection

$$\mathrm{Hom}_{M^\alpha}(v_{M^\alpha \cap P_I}^{M^\alpha}, H^1\mathrm{Ord}_{\overline{P}^\alpha}(v_{P_J}^G)) \hookrightarrow \mathrm{Hom}_{M^\alpha}(i_{M^\alpha \cap P_I}^{M^\alpha}, H^1\mathrm{Ord}_{\overline{P}^\alpha}(v_{P_J}^G))$$

and by [19, Th. 4.4.6], there is an isomorphism

$$\mathrm{Hom}_{M^\alpha}(i_{M^\alpha \cap P_I}^{M^\alpha}, H^1\mathrm{Ord}_{\overline{P}^\alpha}(v_{P_J}^G)) \simeq \mathrm{Hom}_{M_I}(1, \mathrm{Ord}_{M^\alpha \cap \overline{P}_I}(H^1\mathrm{Ord}_{\overline{P}^\alpha}(v_{P_J}^G))).$$

Since $\mathrm{Ord}_{M^\alpha \cap \overline{P}_I}$ is left-exact, it is enough to prove that if $\bar{\varepsilon} \circ \widetilde{\alpha} \neq 1$, then

$$\mathrm{Hom}_{M_I}(1, \mathrm{Ord}_{M^\alpha \cap \overline{P}_I}(H^1\mathrm{Ord}_{\overline{P}^\alpha}(I_{\widehat{w}_J}))) = 0$$

for all $\widehat{w}_J \in \widehat{W}_J$. Using the isomorphism (2.6) and, once again, the results of [33] (or [31, §3.3]), we partially compute $\mathrm{Ord}_{M^\alpha \cap \overline{P}_I}(H^1\mathrm{Ord}_{\overline{P}^\alpha}(I_{\widehat{w}_J}))$. For any $\widehat{w}_J \in \widehat{W}_J$ such that $\widehat{w}_J = s_\alpha w^\alpha$ with $w^\alpha \in W^\alpha$, using the unique decomposition $w^\alpha = \widetilde{w}_I^{-1} w_I$ with $\widetilde{w}_I \in \widetilde{W}_I$ and $w_I \in W_I$, there is a $\overline{B}_{I,w_I}$-equivariant isomorphism

$$\mathrm{Ord}_{M^\alpha \cap \overline{P}_I}(H^1\mathrm{Ord}_{\overline{P}^\alpha}(I_{\widehat{w}_J})) \simeq \begin{cases} \mathrm{LC_c}(\overline{U}_{I,w_I}) \otimes (\varepsilon \circ \widetilde{\alpha}) & \text{if } \widetilde{w}_I = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For any $w_I \in W_I$, we have $\mathrm{LC_c}(\overline{U}_{I,w_I})^{\overline{U}_{I,w_I}} = 0$ if and only if $\overline{U}_{I,w_I} = \{1\}$, i.e. $w_I = w_{I,0}$, hence an isomorphism

$$\mathrm{Hom}_{\overline{B}_{I,w_I}}(1, \mathrm{LC_c}(\overline{U}_{I,w_I}) \otimes (\varepsilon \circ \widetilde{\alpha})) \simeq \begin{cases} \mathrm{Hom}_{\mathscr{Z}}(1, \varepsilon \circ \widetilde{\alpha}) & \text{if } w_I = w_{I,0}, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we see by dévissage that $\mathrm{Hom}_{\mathscr{Z}}(1, \varepsilon \circ \widetilde{\alpha}) = 0$ if $\bar{\varepsilon} \circ \widetilde{\alpha} \neq 1$, hence the result. ☐

Remark 2.7. Assume that $F = \mathbf{Q}_p$ and $\dim \boldsymbol{U}_\alpha = 1$.

(1) We have $\bar{\varepsilon} \circ \widetilde{\alpha} \neq 1$ if $p \geq 5$. More precisely, $\bar{\varepsilon} \circ \alpha = 1$ if and only if either $p = 2$, or $p = 3$ and $\alpha \in 2X^*(\boldsymbol{S})$, in which case the irreducible component of Φ containing α must be of type A_1 or C_l $(l \geq 2)$ with α being the long root.

(2) Assume that $\alpha \in J$ and $\bar{\varepsilon} \circ \widetilde{\alpha} = 1$. We deduce from the proof of Lemma 2.5 that

$$\mathrm{Hom}_{M^\alpha}(v^{M^\alpha}_{M^\alpha \cap P_I}, H^1\mathrm{Ord}_{\overline{P}^\alpha}(v^G_{P_J})) \neq 0$$

only if $s_\alpha w_{I,0} \in \widehat{W}_J$, and the latter condition is equivalent to $J \setminus \{\alpha\} = I \cap \{\alpha\}^\perp$. Actually, we expect the above R-module to be non-zero if and only if $J \setminus \{\alpha\} = I \cap \{\alpha\}^\perp$ and $\Delta \setminus \{\alpha\} = I \cup \{\alpha\}^\perp$. In any case, this R-module is killed by 2 if $\alpha \notin 2X^*(\boldsymbol{S})$, and by 3 or 4 if $\alpha \in 2X^*(\boldsymbol{S})$ (because the same is true for the R-module $\mathrm{Hom}_{\mathscr{Z}}(1, \varepsilon \circ \widetilde{\alpha})$).

2.3.2. *Reduction to the case $J = \varnothing$.* Assume that $J \neq \varnothing$ and let $\alpha \in J$. Taking into account Proposition 2.1, the short exact sequence (2.2) with $J \setminus \{\alpha\}$ instead of I induces an exact sequence

$$0 \to \mathrm{Ext}^1_G(1, v^G_{P_J}) \to \mathrm{Ext}^1_G(1, \mathrm{Ind}^G_{P^\alpha}(v^{M^\alpha}_{P_{J\setminus\{\alpha\}}})) \to \mathrm{Ext}^1_G(1, v^G_{P_{J\setminus\{\alpha\}}}).$$

Using Lemma 2.9 below with $\pi = v^{M^\alpha}_{P_{J\setminus\{\alpha\}}}$, we obtain an exact sequence

$$0 \to \mathrm{Ext}^1_G(1, v^G_{P_J}) \to \mathrm{Ext}^1_{M^\alpha}(1, v^{M^\alpha}_{P_{J\setminus\{\alpha\}}}) \to \mathrm{Ext}^1_G(1, v^G_{P_{J\setminus\{\alpha\}}}). \tag{2.8}$$

Lemma 2.9. *Let π be a smooth M^α-representation. There is an isomorphism*

$$\mathrm{Ext}^1_{M^\alpha}(1, \pi) \simeq \mathrm{Ext}^1_G(1, \mathrm{Ind}^G_{P^\alpha}(\pi)).$$

Proof. By a straightforward generalization of [20, Lemma 4.3.3], there is an isomorphism

$$\mathrm{Ext}^1_{P^\alpha}(1, \pi) \simeq \mathrm{Ext}^1_G(1, \mathrm{Ind}^G_{P^\alpha}(\pi))$$

where P^α acts on π by inflation. Thus it remains to prove that there is an isomorphism

$$\mathrm{Ext}^1_{M^\alpha}(1, \pi) \simeq \mathrm{Ext}^1_{P^\alpha}(1, \pi).$$

By [20, Prop. 2.2.2], such an isomorphism can be rewritten in terms of group cohomology, computed using continuous cochains, as follows:

$$H^1(M^\alpha, \pi) \simeq H^1(P^\alpha, \pi).$$

Using the fact that N^α acts trivially on π, the inflation-restriction exact sequence for continuous group cohomology yields an exact sequence

$$0 \to H^1(M^\alpha, \pi) \to H^1(P^\alpha, \pi) \to \mathrm{Hom}(N^\alpha, \pi)^{M^\alpha}.$$

We prove that the last term is zero. Let $\phi : N^\alpha \to \pi$ be a continuous group homomorphism. The action of $m \in M^\alpha$ on ϕ is given by $m \cdot \phi : n \mapsto m \cdot \phi(m^{-1}nm)$. Since ϕ is continuous, $\ker(\phi)$ is open in N^α. If ϕ is M^α-invariant, then $\ker(\phi) = \ker(m \cdot \phi) = m \ker(\phi) m^{-1}$ for all $m \in M^\alpha$, hence $\ker(\phi) = N^\alpha$, i.e. $\phi = 0$. ☐

We assume from now on that $I = \Delta$, $J = \varnothing$, and $\Delta \neq \varnothing$. The goal of the next paragraphs is to compute the R-module $\mathrm{Ext}^1_G(1, \mathrm{St})$. Using [20, Prop. 2.2.2]*, the latter can be expressed in terms of group cohomology, computed using continuous cochains, as follows:*

$$\mathrm{Ext}^1_G(1, \mathrm{St}) \simeq H^1(G, \mathrm{St}).$$

2.4. **Vanishing and isogenies.** We begin with a few technical lemmas.

Lemma 2.10. *Assume that $\boldsymbol{G} = \boldsymbol{G}_1 \times \boldsymbol{G}_2$ for some connected reductive algebraic F-groups $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ such that $\boldsymbol{G}_1$ has positive relative semisimple rank. Let St_i denote the Steinberg representation of G_i.*

(1) *If $\boldsymbol{G}_2$ also has positive relative semisimple rank, then $\mathrm{Ext}^1_G(1, \mathrm{St}) = 0$.*

(2) *If $\boldsymbol{G}_2$ has relative semisimple rank 0, then there is an isomorphism*

$$\mathrm{Ext}^1_G(1, \mathrm{St}) \xrightarrow{\sim} \mathrm{Ext}^1_{G_1}(1, \mathrm{St}_1).$$

Proof. We write $\boldsymbol{B} = \boldsymbol{B}_1 \times \boldsymbol{B}_2$. There is a G-equivariant isomorphism $i_B^G \simeq i_{B_1}^{G_1} \otimes_R i_{B_2}^{G_2}$ which induces a G-equivariant isomorphism $\mathrm{St} \simeq \mathrm{St}_1 \otimes_R \mathrm{St}_2$. Note that the R-module St_2 is free by [46, Cor. 5.6]. Thus $\mathrm{St}^{G_1} \simeq (\mathrm{St}_1)^{G_1} \otimes_R \mathrm{St}_2 = 0$ by Proposition 2.1. Therefore, using the fact that the R-module St_1 is also free, the inflation-restriction exact sequence for continuous group cohomology yields an isomorphism

$$H^1(G, \mathrm{St}) \xrightarrow{\sim} H^1(G_1, \mathrm{St}_1 \otimes_R (\mathrm{St}_2)^{G_2})$$

and the result follows from Proposition 2.1. □

Lemma 2.11. *Let $\varphi : \boldsymbol{G}' \to \boldsymbol{G}$ be a central isogeny of connected reductive algebraic F-groups. Let* St' *denote the Steinberg representation of G'.*

(1) *If $\mathrm{Ext}^1_{G'}(1, \mathrm{St}') = 0$, then $\mathrm{Ext}^1_G(1, \mathrm{St}) = 0$.*
(2) *The converse holds true if the order of φ is a power of p.*

Proof. Proceeding as in the proof of [1, II.8 Lemma], we see that the R-module St endowed with the action of G' via φ is isomorphic to St'. In particular, $\mathrm{St}^{\varphi(G')} \simeq (\mathrm{St}')^{G'} = 0$ by Proposition 2.1 and φ induces an isomorphism

$$H^1(\varphi(G'), \mathrm{St}) \xrightarrow{\sim} H^1(G', \mathrm{St}').$$

(Here we also use the fact that φ is central and that the center of G' acts trivially on St'.) Therefore, the inflation-restriction exact sequence for continuous group cohomology yields an isomorphism

$$H^1(G, \mathrm{St}) \xrightarrow{\sim} H^1(G', \mathrm{St}')^{G/\varphi(G')},$$

hence (1). Moreover, if the order of φ is a power of p, then $G/\varphi(G')$ is a pro-p group acting smoothly on $H^1(G', \mathrm{St}')$, hence (2). □

2.5. Comparison of Ext^1 for G and for $\overline{B}$.

2.5.1. *Construction of a map.* We construct a map

$$\mathrm{Ext}^1_G(1, \mathrm{St}) \to \mathrm{Ext}^1_{\mathscr{Z}}(1, 1). \tag{2.12}$$

We deduce from the Bruhat filtrations that there is a $\overline{B}$-equivariant isomorphism $\jmath : \mathrm{LC_c}(\overline{U}) \xrightarrow{\sim} \mathrm{St}$ (see the proof of Lemma 2.5 and use the fact that $\widehat{W}_\varnothing = \{1\}$), which is the composite

$$\mathrm{LC_c}(\overline{U}) \hookrightarrow i_B^G \twoheadrightarrow \mathrm{St} \tag{2.13}$$

where the first map is induced by the open immersion $\overline{U} \hookrightarrow B\backslash G$ and the second map is the G-equivariant surjection. The evaluation at $1 \in \overline{U}$ yields a $\mathscr{Z}$-equivariant surjection $\mathrm{ev}_1 : \mathrm{LC_c}(\overline{U}) \twoheadrightarrow 1$. The $\mathscr{Z}$-equivariant surjection $\mathrm{ev}_1 \circ \jmath^{-1} : \mathrm{St} \twoheadrightarrow 1$ induces a map

$$\mathrm{Ext}^1_{\overline{B}}(1, \mathrm{St}) \to \mathrm{Ext}^1_{\mathscr{Z}}(1, 1). \tag{2.14}$$

Lemma 2.15. (1) *Any extension in the image of* (2.14) *is trivial on Z_α for all $\alpha \in \Delta$.*
(2) *If $|\Delta| > 2$ or $\Delta = \{\alpha, \beta\}$ with $\alpha \perp \beta$, then* (2.14) *is zero.*

Proof. Let $I \subset \Delta$ and St_I denote the Steinberg representation of M_I. Since ev_1 factors through the $\overline{B}_I$-equivariant surjection $\mathrm{LC_c}(\overline{U}) \twoheadrightarrow \mathrm{LC_c}(\overline{U}_I)$ given by the restriction to $\overline{U}_I$, (2.14) factors through the map

$$\mathrm{Ext}^1_{\overline{B}_I}(1, \mathrm{St}_I) \to \mathrm{Ext}^1_{\mathscr{Z}}(1, 1) \tag{2.16}$$

induced by the $\overline{B}_I$-equivariant isomorphism $\mathrm{St}_I \xrightarrow{\sim} \mathrm{LC_c}(\overline{U}_I)$ and the $\mathscr{Z}$-equivariant surjection $\mathrm{LC_c}(\overline{U}_I) \twoheadrightarrow 1$ given by the evaluation at $1 \in \overline{U}_I$. Since $\mathrm{Hom}_{\overline{B}_I}(1, \mathrm{St}_I) = 0$ (as can be seen from the previous isomorphism), any extension in the source of (2.16) is trivial on Z_I, hence (1) with $I = \{\alpha\}$. Now assume that there exist $\alpha, \beta \in \Delta$ such that $\alpha \perp \beta$ (e.g. $|\Delta| > 2$). Proceeding as in the proof of Lemmas 2.10(1) and 2.11, we see that the source of (2.16) with $I = \{\alpha, \beta\}$ is zero, hence (2). □

Finally, we define (2.12) as the map obtained by composing (2.14) with the restriction map

$$\operatorname{Ext}^1_G(1,\mathrm{St}) \to \operatorname{Ext}^1_{\overline{B}}(1,\mathrm{St}). \tag{2.17}$$

2.5.2. *Injectivity of the map.* First, the map (2.17) is injective by a straightforward generalization of [20, Lemma 4.3.5]. In order to prove that the map (2.14) is also injective, we generalize the results of [20, §A.2]. There is a $\overline{B}$-equivariant isomorphism

$$\operatorname{Ind}_{\mathscr{Z}}^{\overline{B}}(1) \simeq \mathrm{LC_u}(\overline{U}) \tag{2.18}$$

and a long exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_{\overline{B}}(1,\mathrm{LC_c}(\overline{U})) \to \operatorname{Hom}_{\overline{B}}(1,\mathrm{LC_u}(\overline{U})) \to \operatorname{Hom}_{\overline{B}}(1,\mathrm{LC_u}(\overline{U})/\mathrm{LC_c}(\overline{U})) \\ &\to \operatorname{Ext}^1_{\overline{B}}(1,\mathrm{LC_c}(\overline{U})) \to \operatorname{Ext}^1_{\overline{B}}(1,\mathrm{LC_u}(\overline{U})). \end{aligned}$$

Clearly $\operatorname{Hom}_{\overline{B}}(1,\mathrm{LC_c}(\overline{U}))=0$, while Frobenius reciprocity and (2.18) induce isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\overline{B}}(1,\mathrm{LC_u}(\overline{U})) &\simeq \operatorname{Hom}_{\mathscr{Z}}(1,1), \\ \operatorname{Ext}^1_{\overline{B}}(1,\mathrm{LC_u}(\overline{U})) &\simeq \operatorname{Ext}^1_{\mathscr{Z}}(1,1). \end{aligned}$$

Moreover, using the latter isomorphism and the isomorphism $\jmath$ (which is the composite (2.13)), the last map of the above long exact sequence can be identified with (2.14). Finally, by a straightforward generalization of [20, Lemma 4.3.4], there is an isomorphism

$$\operatorname{Hom}_{\overline{B}}(1,\mathrm{LC_u}(\overline{U})/\mathrm{LC_c}(\overline{U})) \simeq \operatorname{Hom}_{\mathscr{Z}}(1,\mathrm{LC_u}(\overline{U})/\mathrm{LC_c}(\overline{U})).$$

Therefore, the above long exact sequence can be rewritten as

$$0 \to \operatorname{Hom}_{\mathscr{Z}}(1,1) \to \operatorname{Hom}_{\mathscr{Z}}(1,\mathrm{LC_u}(\overline{U})/\mathrm{LC_c}(\overline{U})) \to \operatorname{Ext}^1_{\overline{B}}(1,\mathrm{St}) \xrightarrow{(2.14)} \operatorname{Ext}^1_{\mathscr{Z}}(1,1). \tag{2.19}$$

Lemma 2.20. *Assume that the adjoint action of $\mathscr{Z}$ on $\overline{U}\setminus\{1\}$ is transitive or that $|\Delta|>1$. Then the R-module $\operatorname{Hom}_{\mathscr{Z}}(1,\mathrm{LC_u}(\overline{U})/\mathrm{LC_c}(\overline{U}))$ is free of rank* 1.

Proof. Let $f\in\mathrm{LC_u}(\overline{U})$ such that $z\cdot f-f\in\mathrm{LC_c}(\overline{U})$ for all $z\in\mathscr{Z}$. We will prove that $f\in R+\mathrm{LC_c}(\overline{U})$, where we identify R with the R-submodule of $\mathrm{LC_u}(\overline{U})$ consisting of the constant functions on $\overline{U}$.

We fix a compact open subgroup $\overline{U}_0\subset\overline{U}$ and an element $z_+\in Z_{\mathscr{Z}}$ strictly contracting $\overline{U}_0$, i.e. such that $z_+\overline{U}_0z_+^{-1}\subset\overline{U}_0$ and $\bigcap_{i\geq1}z_+^i\overline{U}_0z_+^{-i}=\{1\}$ (or equivalently, $\bigcup_{i\geq1}z_+^{-i}\overline{U}_0z_+^i=\overline{U}$). We have a decomposition of $\overline{U}$ into a disjoint union of open subsets

$$\overline{U}=\overline{U}_0\sqcup\bigsqcup_{i\geq1}z_+^{-i}(\overline{U}_0\setminus z_+\overline{U}_0z_+^{-1})z_+^i.$$

Correspondingly, we can write

$$f=f_0+\sum_{i\geq1}z_+^{-i}\cdot f_i$$

with $\operatorname{supp}(f_0)\subset\overline{U}_0$ and $\operatorname{supp}(f_i)\subset\overline{U}_0\setminus z_+\overline{U}_0z_+^{-1}$ for all $i\geq1$. We have

$$z_+\cdot f-f=z_+\cdot f_0-f_0+f_1+\sum_{i\geq1}z_+^{-i}\cdot(f_{i+1}-f_i).$$

Since $z_+\cdot f-f\in\mathrm{LC_c}(\overline{U})$ and $\operatorname{supp}(z_+^{-i}\cdot(f_{i+1}-f_i))\subset z_+^{-i}(\overline{U}_0\setminus z_+\overline{U}_0z_+^{-1})z_+^i$ for all $i\geq1$, there exists $j\geq1$ such that $f_{i+1}=f_i$ for all $i\geq j$. Thus we can rewrite

$$f=f_0+\sum_{i\geq1}z_+^{-i}\cdot f_\infty$$

with $f_0\in\mathrm{LC_c}(\overline{U})$ and $\operatorname{supp}(f_\infty)\subset\overline{U}_0\setminus z_+\overline{U}_0z_+^{-1}$. Note that $f\in R+\mathrm{LC_c}(\overline{U})$ if and only if f_∞ is constant on $\overline{U}_0\setminus z_+\overline{U}_0z_+^{-1}$.

We prove a key identity. Let $\bar{u} \in \overline{U}_0 \setminus z_+\overline{U}_0 z_+^{-1}$ and $z \in \mathscr{Z}$ such that $z\bar{u}z^{-1} \in \overline{U}_0 \setminus z_+\overline{U}_0 z_+^{-1}$. For any $j \geq 0$ large enough so that $z_+^{-j}\bar{u}z_+^j \notin \operatorname{supp}(z^{-1}\cdot f - f)$ and $z_+^{-j}\bar{u}z_+^j \notin \operatorname{supp}(z^{-1}\cdot f_0 - f_0)$, we have

$$\begin{aligned}
0 &= (z^{-1}\cdot f - f)(z_+^{-j}\bar{u}z_+^j) \\
&= (z^{-1}\cdot f_0 - f_0)(z_+^{-j}\bar{u}z_+^j) + \sum_{i\geq 1}(z^{-1}\cdot f_\infty - f_\infty)(z_+^{i-j}\bar{u}z_+^{j-i}) \\
&= \sum_{i\geq 1} f_\infty(z_+^{i-j}(z\bar{u}z^{-1})z_+^{j-i}) - \sum_{i\geq 1} f_\infty(z_+^{i-j}\bar{u}z_+^{j-i}) \\
&= f_\infty(z\bar{u}z^{-1}) - f_\infty(\bar{u}),
\end{aligned}$$

the last equality resulting from the fact that $\bar{u} \in \overline{U}_0 \setminus z_+\overline{U}_0 z_+^{-1}$ and $z\bar{u}z^{-1} \in \overline{U}_0 \setminus z_+\overline{U}_0 z_+^{-1}$ whereas $\operatorname{supp}(f_\infty) \subset \overline{U}_0 \setminus z_+\overline{U}_0 z_+^{-1}$, so that $f_\infty(z_+^{i-j}(z\bar{u}z^{-1})z_+^{j-i}) = f_\infty(z_+^{i-j}\bar{u}z_+^{j-i}) = 0$ if $j \neq i$. Therefore,

$$f_\infty(z\bar{u}z^{-1}) = f_\infty(\bar{u}). \tag{2.21}$$

In particular, we deduce that f_∞ is constant on $\overline{U}_0 \setminus z_+\overline{U}_0 z_+^{-1}$ if the adjoint action of $\mathscr{Z}$ on $\overline{U} \setminus \{1\}$ is transitive.

Now let $\alpha \in \Delta$. We set $\overline{U}_{\alpha,0} = \overline{U}_\alpha \cap \overline{U}_0$ and $\overline{N}_{\alpha,0} = \overline{N}_\alpha \cap \overline{U}_0$. Replacing $\overline{U}_0$ by $\overline{U}_{\alpha,0}\overline{N}_{\alpha,0}$, we can assume that $\overline{U}_0 = \overline{U}_{\alpha,0}\overline{N}_{\alpha,0}$, hence $z_+\overline{U}_0 z_+^{-1} = (z_+\overline{U}_{\alpha,0}z_+^{-1})(z_+\overline{N}_{\alpha,0}z_+^{-1})$, and we fix an element $z_{\alpha,+} \in Z_\alpha$ strictly contracting $\overline{N}_{\alpha,0}$. Let $\bar{u} \in \overline{U}_0 \setminus z_+\overline{U}_0 z_+^{-1}$. We write $\bar{u} = \bar{u}_\alpha \bar{n}_\alpha$ with $\bar{u}_\alpha \in \overline{U}_{\alpha,0}$ and $\bar{n}_\alpha \in \overline{N}_{\alpha,0}$. Let $i \geq 0$ such that $z_+^{-i}\bar{u}_\alpha z_+^i \in \overline{U}_{\alpha,0} \setminus z_+\overline{U}_{\alpha,0}z_+^{-1}$. For any $j \geq 0$ large enough so that $z_{\alpha,+}^j z_+^{-i}\bar{n}_\alpha z_+^i z_{\alpha,+}^{-j} \in \overline{N}_{\alpha,0}$, we have

$$z_{\alpha,+}^j z_+^{-i}\bar{u}z_+^i z_{\alpha,+}^{-j} = (z_+^{-i}\bar{u}_\alpha z_+^i)(z_{\alpha,+}^j z_+^{-i}\bar{n}_\alpha z_+^i z_{\alpha,+}^{-j}) \in \overline{U}_0 \setminus z_+\overline{U}_0 z_+^{-1},$$

hence using (2.21) we obtain

$$f_\infty(\bar{u}) = f_\infty((z_+^{-i}\bar{u}_\alpha z_+^i)(z_{\alpha,+}^j z_+^{-i}\bar{n}_\alpha z_+^i z_{\alpha,+}^{-j})).$$

Since f_∞ is locally constant, we deduce by passing to the limit as $j \to +\infty$ that

$$f_\infty(\bar{u}) = f_\infty(z_+^{-i}\bar{u}_\alpha z_+^i). \tag{2.22}$$

Therefore, f_∞ is constant on $\overline{U}_0 \setminus z_+\overline{U}_0 z_+^{-1}$ if and only if it is constant on $\overline{U}_{\alpha,0} \setminus z_+\overline{U}_{\alpha,0}z_+^{-1}$.

Finally, assume that $|\Delta| > 1$ and fix $\bar{n}_\alpha \in \overline{N}_{\alpha,0} \setminus z_+\overline{N}_{\alpha,0}z_+^{-1}$. For any $\bar{u}_\alpha \in \overline{U}_{\alpha,0} \setminus z_+\overline{U}_{\alpha,0}z_+^{-1}$ and for any $i \geq 0$, we have $z_+^i\bar{u}_\alpha z_+^{-i}\bar{n}_\alpha \in \overline{U}_0 \setminus z_+\overline{U}_0 z_+^{-1}$, hence using (2.22) we obtain

$$f_\infty(z_+^i\bar{u}_\alpha z_+^{-i}\bar{n}_\alpha) = f_\infty(\bar{u}_\alpha).$$

Since f_∞ is locally constant, we deduce by passing to the limit as $i \to +\infty$ that

$$f_\infty(\bar{n}_\alpha) = f_\infty(\bar{u}_\alpha).$$

Therefore, f_∞ is constant on $\overline{U}_{\alpha,0} \setminus z_+\overline{U}_{\alpha,0}z_+^{-1}$. □

Using the exact sequence (2.19), we deduce from Lemma 2.20 the following result.

Proposition 2.23. *Assume that the adjoint action of $\mathscr{Z}$ on $\overline{U} \setminus \{1\}$ is transitive or that $|\Delta| > 1$. Then* (2.12) *is injective.*

Remark 2.24. If $\Delta = \{\alpha\}$ and the adjoint action of $\mathscr{Z}$ on $\overline{U} \setminus \{1\}$ is not transitive, then (2.14) need not be injective. For example, if $G = \mathrm{SL}_2(F)$, then the R-module in Lemma 2.20 is free of rank $|F^*/F^{*2}| > 1$ so that (2.14) is not injective. However, if $\Delta = \{\alpha\}$ and $p \neq 2$, then one can prove that (2.12) is still injective under the weaker condition that for all $\bar{u} \in \overline{U} \setminus \{1\}$ there exists $z \in \mathscr{Z}$ such that $z\bar{u}z^{-1} = \bar{u}^{-1}$. For example, if $G = \mathrm{SL}_2(F)$ and $p \neq 2$, then (2.12) is injective if $-1 \in F^{*2}$, i.e. if $q \equiv 1 \bmod 4$, where q denotes the cardinal of the residue field of F.

2.5.3. *Image of the map.* The R-module $\mathrm{Ext}^1_{\mathscr{Z}}(1,1)$ is endowed with the action of W induced by the action of $\mathscr{N}$ on $\mathscr{Z}$ by conjugation. The corresponding action of W on the R-module $\mathrm{Hom}(\mathscr{Z},R)$ via the isomorphism

$$\mathrm{Ext}^1_{\mathscr{Z}}(1,1)\simeq \mathrm{Hom}(\mathscr{Z},R)$$

is given by $w\cdot\phi : z\mapsto \phi(n^{-1}zn)$ where $n\in\mathscr{N}$ is a representative of w. We also write $\phi^w = w^{-1}\cdot\phi$. When $w=s_\alpha$ for some $\alpha\in\Delta$, we rather write ϕ^α. We let $\epsilon : W\to\{\pm 1\}$ be the character defined by $\epsilon(w)=(-1)^{l_w}$ and we write $\mathrm{Ext}^1_{\mathscr{Z}}(1,1)^\epsilon$ for the ϵ-isotypic component of $\mathrm{Ext}^1_{\mathscr{Z}}(1,1)$, that is the R-submodule consisting of those ϕ such that $w\cdot\phi=\epsilon(w)\phi$ for all $w\in W$. When $\Delta=\{\alpha\}$, we rather write $\mathrm{Ext}^1_{\mathscr{Z}}(1,1)^{s_\alpha=-1}$.

Lemma 2.25. *The image of* (2.12) *lies in* $\mathrm{Ext}^1_{\mathscr{Z}}(1,1)^\epsilon$.

Proof. Using [20, Prop. 2.2.2], we can rewrite the map (2.12) in terms of group cohomology, computed using continuous cochains, as follows:

$$H^1(G,\mathrm{St})\to H^1(\mathscr{Z},1)\simeq\mathrm{Hom}(\mathscr{Z},R).$$

Explicitly, this map sends a continuous cochain $\phi : G\to\mathrm{St}$ to the continuous group homomorphism $z\mapsto \jmath^{-1}(\phi(z))(1)$ where the isomorphism $\jmath$ is the composite (2.13). It is enough to prove that for all $\alpha\in\Delta$, s_α acts by multiplication by -1 on the image of this map.

Let $\alpha\in\Delta$ and $n_\alpha\in\mathscr{N}$ be a representative of s_α. If $\phi : G\to\mathrm{St}$ is a continuous cochain, then ϕ and $n_\alpha\cdot\phi : g\mapsto n_\alpha\cdot\phi(n_\alpha^{-1}gn_\alpha)$ are cohomologous since $n_\alpha\in G$, thus their images in $\mathrm{Hom}(\mathscr{Z},R)$ are equal, i.e. $\jmath^{-1}(\phi(z))(1)=\jmath^{-1}(n_\alpha\cdot\phi(n_\alpha^{-1}zn_\alpha))(1)$ for all $z\in\mathscr{Z}$. Therefore, it is enough to prove that $\jmath^{-1}(n_\alpha\cdot v)(1)=-\jmath^{-1}(v)(1)$ for all $v\in\mathrm{St}$.

Let $v\in\mathrm{St}$ and f (resp. f_α) be the unique lift of v (resp. $n_\alpha\cdot v$) in i_B^G with support in $B\overline{U}$. Since $n_\alpha\cdot f$ and f_α have the same image in St (that is $n_\alpha\cdot v$), we have $n_\alpha\cdot f-f_\alpha\in\sum_{I\neq\varnothing} i^G_{P_I}$. The support of $n_\alpha\cdot f$ is contained in $B\overline{P}_\alpha$, hence $n_\alpha\cdot f-f_\alpha\in\text{c-Ind}_B^{B\overline{P}_\alpha}1$. By [2, Lemma 7.8], we have

$$\Big(\sum_{I\neq\varnothing} i^G_{P_I}\Big)\cap\text{c-Ind}_B^{B\overline{P}_\alpha}1=\sum_{I\neq\varnothing}\big(i^G_{P_I}\cap\text{c-Ind}_B^{B\overline{P}_\alpha}1\big).$$

Since $B\overline{P}_\alpha=P_\alpha\overline{B}$, we deduce from [2, Prop. 7.11] that, for $I\neq\varnothing$, we have

$$i^G_{P_I}\cap\text{c-Ind}_B^{B\overline{P}_\alpha}1=\begin{cases}\text{c-Ind}_{P_\alpha}^{P_\alpha\overline{B}}1 & \text{if } P=P_\alpha,\\ 0 & \text{otherwise.}\end{cases}$$

Thus $n_\alpha\cdot f-f_\alpha\in\text{c-Ind}_{P_\alpha}^{P_\alpha\overline{B}}1$. Now we compute:

$$\jmath^{-1}(v)(1)=f(1)=(n_\alpha\cdot f-f_\alpha)(n_\alpha^{-1})=(n_\alpha\cdot f-f_\alpha)(1)=-f_\alpha(1)=-\jmath^{-1}(n_\alpha\cdot v)(1),$$

hence the result. □

We generalize the proof of [20, Prop. 4.3.22(2)]. We consider the composite

$$\mathrm{Ext}^1_{\mathscr{Z}}(1,1)\hookrightarrow\mathrm{Ext}^1_G(i_B^G,i_B^G)\to\mathrm{Ext}^1_G(1,\mathrm{St})\xrightarrow{(2.12)}\mathrm{Ext}^1_{\mathscr{Z}}(1,1)^\epsilon \tag{2.26}$$

where the first map is induced by the exact functor Ind_B^G, the second map is induced by the G-equivariant injection $1\hookrightarrow i_B^G$ and the G-equivariant surjection $i_B^G\twoheadrightarrow\mathrm{St}$, and the third map is (2.12) (which takes values in $\mathrm{Ext}^1_{\mathscr{Z}}(1,1)^\epsilon$ by Lemma 2.25).

Lemma 2.27. *Assume that* $\Delta=\{\alpha\}$. *Then* (2.26) *is given by* $\phi\mapsto\phi-\phi^\alpha$.

Proof. First, we consider the $\mathscr{Z}$-equivariant surjection

$$\pi : i_B^G\twoheadrightarrow\mathrm{St}\xrightarrow{\sim}\mathrm{LC_c}(\overline{U})\twoheadrightarrow 1$$

where the first map is the G-equivariant surjection, the second map is the G-equivariant isomorphism $\jmath^{-1}$ (which is the inverse of the composite (2.13)), and the third map is the evaluation at $1\in\overline{U}$ (which is $\mathscr{Z}$-equivariant). Let $n_\alpha \in \mathscr{N}$ be a representative of s_α. For any $f \in i_B^G$, the function $f - f(n_\alpha) \in i_B^G$ has the same image as f in St and $\operatorname{supp}(f - f(n_\alpha)) \subset B\overline{U}$. Therefore, $\pi(f) = f(1) - f(n_\alpha)$ for all $f \in i_B^G$.

Now let $\phi : \mathscr{Z} \to R$ be a continuous group homomorphism. We let E be the corresponding extension of 1 by 1, i.e. $E = Re_1 \oplus Re_2$ as an R-module and the smooth action of $\mathscr{Z}$ is given by $z \cdot e_1 = e_1$ and $z \cdot e_2 = e_2 + \phi(z)e_1$. We consider the smooth G-representation $\operatorname{Ind}_B^G(E)$, which is an extension of i_B^G by i_B^G. Any $f \in \operatorname{Ind}_B^G(E)$ can be uniquely written $f = f_1e_1 + f_2e_2$ with $f_1, f_2 \in \mathrm{LC}(G)$ satisfying $f_1(zug) = f_1(g) + \phi(z)f_2(g)$ and $f_2(zug) = f_2(g)$ for all $z \in \mathscr{Z}$, $u \in U$, and $g \in G$. We let I denote the extension of 1 by i_B^G obtained from $\operatorname{Ind}_B^G(E)$ by pullback along the G-equivariant injection $1 \hookrightarrow i_B^G$, i.e. I is the G-subrepresentation of $\operatorname{Ind}_B^G(E)$ consisting of those functions $f = f_1e_1 + f_2e_2$ with f_2 constant. The image of E under (2.26) is the extension E' of 1 by 1 obtained from I by pushforward along the $\mathscr{Z}$-equivariant surjection π. We fix $f \in I$ such that $f = f_1e_1 + e_2$, so that $f_1(zug) = f_1(g) + \phi(z)$ for all $z \in \mathscr{Z}$, $u \in U$, and $g \in G$. Therefore, the continuous group homomorphism $\phi' : \mathscr{Z} \to R$ corresponding to E' is given by

$$\begin{aligned}
\phi'(z) &= \pi(z \cdot f_1 - f_1) \\
&= (z \cdot f_1 - f_1)(1) - (z \cdot f_1 - f_1)(n_\alpha) \\
&= (f_1(z) - f_1(1)) - (f_1((n_\alpha z n_\alpha^{-1})n_\alpha) - f_1(n_\alpha)) \\
&= \phi(z) - \phi(n_\alpha z n_\alpha^{-1}),
\end{aligned}$$

i.e. $\phi' = \phi - \phi^\alpha$. □

Remark 2.28. In general, one can prove that the map π in the proof of Lemma 2.27 is given by $\pi(f) = \sum_{w \in W} \epsilon(w) f(n_w)$, where n_w is a representative of w, for all $f \in i_B^G$. Therefore, (2.26) is given by $\phi \mapsto \sum_{w \in W} \epsilon(w)\phi^w$.

We deduce from Lemma 2.27 that (2.26) is surjective if $\Delta = \{\alpha\}$ and $p \neq 2$. Since (2.26) factors through (2.12), we obtain the following result.

Proposition 2.29. *Assume that* $\Delta = \{\alpha\}$ *and* $p \neq 2$*. Then the image of* (2.12) *is* $\operatorname{Ext}^1_{\mathscr{Z}}(1,1)^{s_\alpha = -1}$.

Remark 2.30. Assume that $\Delta = \{\alpha\}$ and $p = 2$. Using Lemma 2.27, we see that (2.26) is surjective if $G = \mathrm{GL}_2(D)$ for some division algebra D over F, so that the image of (2.12) is $\operatorname{Ext}^1_{\mathscr{Z}}(1,1)^{s_\alpha = -1}$ in this case too. On the other hand, we see that (2.26) is zero if $G = \mathrm{SL}_2(F)$ and $\operatorname{char}(R) = 2$.

2.6. A vanishing result.

This paragraph is devoted to the fairly technical proof of the following vanishing result.

Proposition 2.31. *Assume that* $\boldsymbol{G}$ *is absolutely almost-simple,* $|\Delta| = 2$*, and* $p \neq 2$*. If* $p \nmid |\boldsymbol{Z_G}|$*, then* (2.12) *is zero.*

Thanks to Lemma 2.25, this result is a consequence of the short exact sequence (2.36) and of Lemmas 2.37 and 2.39 below.

Let $\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}$ be the simply connected covering of the derived subgroup $\boldsymbol{\mathscr{Z}}_{\mathrm{der}}$ of $\boldsymbol{\mathscr{Z}}$. The action of $\boldsymbol{\mathscr{N}}$ on $\boldsymbol{\mathscr{Z}}$ by conjugation stabilizes $\boldsymbol{\mathscr{Z}}_{\mathrm{der}}$ and $\boldsymbol{Z_{\mathscr{Z}}}$. The induced action of $\boldsymbol{\mathscr{N}}$ on $\boldsymbol{\mathscr{Z}}_{\mathrm{der}}$ lifts uniquely to $\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}$ and stabilizes $\boldsymbol{Z_{\mathscr{Z}_{\mathrm{sc}}}}$. Note that the action of $\boldsymbol{\mathscr{N}}$ on $\boldsymbol{Z_{\mathscr{Z}}}$ and $\boldsymbol{Z_{\mathscr{Z}_{\mathrm{sc}}}}$ factors through W. Moreover, the morphisms of the canonical short exact sequence of algebraic F-groups

$$1 \to \boldsymbol{Z_{\mathscr{Z}_{\mathrm{sc}}}} \to \boldsymbol{\mathscr{Z}}_{\mathrm{sc}} \times \boldsymbol{Z_{\mathscr{Z}}} \to \boldsymbol{\mathscr{Z}} \to 1 \tag{2.32}$$

are $\boldsymbol{\mathscr{N}}$-equivariant.

Recall that $\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}$ is the direct product of its almost-simple factors, each of which is a simply connected almost-simple anisotropic algebraic F-groups, hence an isomorphism

$$\mathscr{Z}_{\mathrm{sc}} \simeq \mathrm{SL}_1(D_1) \times \cdots \times \mathrm{SL}_1(D_r) \tag{2.33}$$

for some division algebras $D_1, \dots, D_r$ over F and $H^1(F, \boldsymbol{\mathscr{Z}}_{\mathrm{sc}}) = 0$. These results are due to Kneser [40, 41] when $\mathrm{char}(F) = 0$ and Bruhat-Tits [8] in general. Thus passing to F-points in (2.32) yields an exact sequence of topological $\mathscr{N}$-modules

$$1 \to Z_{\mathscr{Z}_{\mathrm{sc}}} \to \mathscr{Z}_{\mathrm{sc}} \times Z_{\mathscr{Z}} \to \mathscr{Z} \to H^1(F, \boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}}) \to H^1(F, \boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}}). \tag{2.34}$$

By a result of Riehm [61], the abelianization of $\mathrm{SL}_1(D_i)$ is a finite group of prime-to-p order (see the corollary to Theorem 21 and Theorem 7(iii) in loc. cit.), hence $\mathrm{Hom}(\mathscr{Z}_{\mathrm{sc}}, R) = 0$. Therefore, (2.34) induces an exact sequence

$$0 \to \mathrm{Hom}(H_{\mathscr{Z}}, R) \to \mathrm{Hom}(\mathscr{Z}, R) \to \mathrm{Hom}(Z_{\mathscr{Z}}, R) \tag{2.35}$$

where $H_{\mathscr{Z}}$ is the topological abelian group defined by

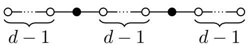
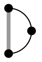
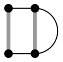
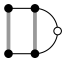
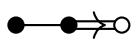
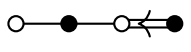
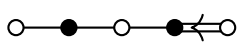
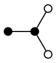
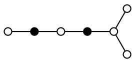
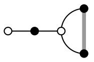
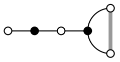
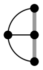
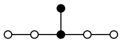
$$H_{\mathscr{Z}} = \ker(H^1(F, \boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}}) \to H^1(F, \boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}})).$$

Note that $Z_{\mathscr{Z}}$ and $H_{\mathscr{Z}}$ are both topological W-modules. Taking the ϵ-isotypic components in (2.35) yields an exact sequence

$$0 \to \mathrm{Hom}(H_{\mathscr{Z}}, R)^\epsilon \to \mathrm{Hom}(\mathscr{Z}, R)^\epsilon \to \mathrm{Hom}(Z_{\mathscr{Z}}, R)^\epsilon. \tag{2.36}$$

We will assume for the rest of this paragraph that $\boldsymbol{G}$ is absolutely almost-simple, $|\Delta| = 2$, and $p \neq 2$. Moreover, we will write $\Delta = \{\alpha, \beta\}$. In Table 1 below, we list the possible Tits indices [65] *for $\boldsymbol{G}$ (extracted from Table 2 in loc. cit.) with the corresponding groups $\boldsymbol{Z}_{\boldsymbol{G}_{\mathrm{sc}}, \overline{F}}$ and $\boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}, \overline{F}}$.*

TABLE 1. Tits indices of relative rank 2 over a local field

Name	Index	$\boldsymbol{Z}_{\boldsymbol{G}_{\mathrm{sc}}, \overline{F}}$	$\boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}, \overline{F}}$
$^1\mathrm{A}^{(d)}_{3d-1,2}$		$\boldsymbol{\mu}_{3d}$	$\boldsymbol{\mu}_d \times \boldsymbol{\mu}_d \times \boldsymbol{\mu}_d$
$^2\mathrm{A}^{(1)}_{3,2}$		$\boldsymbol{\mu}_4$	1
$^2\mathrm{A}^{(1)}_{4,2}$		$\boldsymbol{\mu}_5$	1
$^2\mathrm{A}^{(1)}_{5,2}$		$\boldsymbol{\mu}_6$	$\boldsymbol{\mu}_2$
$\mathrm{B}_{2,2}$		$\boldsymbol{\mu}_2$	1
$\mathrm{B}_{3,2}$		$\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_2$
$\mathrm{C}^{(2)}_{4,2}$		$\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$
$\mathrm{C}^{(2)}_{5,2}$		$\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$
$^1\mathrm{D}^{(1)}_{4,2}$		$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$	$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$
$^1\mathrm{D}^{(2)}_{7,2}$		$\boldsymbol{\mu}_4$	$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_4$
$^2\mathrm{D}^{(2)}_{5,2}$		$\boldsymbol{\mu}_4$	$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$
$^2\mathrm{D}^{(2)}_{6,2}$		$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$	$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$
$^3\mathrm{D}^2_{4,2}$ $^6\mathrm{D}^2_{4,2}$		$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$	1
$^1\mathrm{E}^{16}_{6,2}$		$\boldsymbol{\mu}_3$	$\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$
$\mathrm{G}^0_{2,2}$		1	1

Lemma 2.37. *Assume that $p \nmid |\boldsymbol{Z}_{\boldsymbol{G}}|$. Then* $\mathrm{Hom}(Z_{\mathscr{Z}}, R)^\epsilon = 0$.

Proof. We claim that there is a short exact sequence of algebraic F-groups of multiplicative type

$$1 \to \boldsymbol{Z}_{\boldsymbol{G}} \to \boldsymbol{Z}_\alpha \times \boldsymbol{Z}_\beta \to \boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}} \to 1. \tag{2.38}$$

Let X be the group of characters of a maximal torus of $\boldsymbol{G}_{\overline{F}}$ containing $\boldsymbol{S}_{\overline{F}}$ and Δ^{abs} be the set of absolute simple roots in X relative to a Borel subgroup $\boldsymbol{G}_{\overline{F}}$ contained in $\boldsymbol{B}_{\overline{F}}$. The restriction map

$$r : X \to X^*(\boldsymbol{S}_{\overline{F}}) \simeq X^*(\boldsymbol{S})$$

induces a map

$$r : \Delta^{\mathrm{abs}} \to \{\alpha, \beta, 0\}.$$

We set $\Delta^{\mathrm{abs}}_\alpha = r^{-1}(\{\alpha, 0\})$, $\Delta^{\mathrm{abs}}_\beta = r^{-1}(\{\beta, 0\})$, and $\Delta^{\mathrm{abs}}_0 = r^{-1}(\{0\})$. These are the subsets of absolute simple roots corresponding to the Levi subgroups $\boldsymbol{M}_{\alpha,\overline{F}}$, $\boldsymbol{M}_{\beta,\overline{F}}$, and $\boldsymbol{\mathscr{Z}}_{\overline{F}}$ respectively. We set $X_\alpha = \mathbf{Z}[\Delta^{\mathrm{abs}}_\alpha]$ and $X_\beta = \mathbf{Z}[\Delta^{\mathrm{abs}}_\beta]$. Note that $X_\alpha \cap X_\beta = \mathbf{Z}[\Delta^{\mathrm{abs}}_0]$ and $X_\alpha + X_\beta = \mathbf{Z}[\Delta^{\mathrm{abs}}]$. Therefore, (2.38) corresponds to the short exact sequence of $\mathscr{G}_F$-modules

$$0 \to X/(X_\alpha \cap X_\beta) \to (X/X_\alpha) \oplus (X/X_\beta) \to X/(X_\alpha + X_\beta) \to 0$$

via the duality between algebraic F-groups of multiplicative type and finitely generated abelian groups endowed with a continuous action of $\mathscr{G}_F$ (see [47, §12g]).

Now let $\phi : Z_{\mathscr{Z}} \to R$ be a continuous group homomorphism such that $s_\alpha \cdot \phi = s_\beta \cdot \phi = -\phi$. Passing to F-points in (2.38) yields an exact sequence of topological abelian groups

$$1 \to Z_G \to Z_\alpha \times Z_\beta \to Z_{\mathscr{Z}} \to H^1(F, \boldsymbol{Z}_{\boldsymbol{G}}) \to H^1(F, \boldsymbol{Z}_\alpha) \times H^1(F, \boldsymbol{Z}_\beta).$$

The actions of s_α and s_β on Z_α and Z_β respectively are trivial. Since $p \neq 2$, we deduce that ϕ is trivial on Z_α and Z_β. Therefore, ϕ factors through $\ker(H^1(F, \boldsymbol{Z}_{\boldsymbol{G}}) \to H^1(F, \boldsymbol{Z}_\alpha) \times H^1(F, \boldsymbol{Z}_\beta))$, which is $|\boldsymbol{Z}_{\boldsymbol{G}}|$-torsion. We conclude that $\phi = 0$ since $p \nmid |\boldsymbol{Z}_{\boldsymbol{G}}|$. ☐

Lemma 2.39. *If $\boldsymbol{G}$ is of type ${}^1\mathrm{A}^{(d)}_{3d-1,2}$ and $p = 3$ assume that $3 \nmid |\boldsymbol{Z}_{\boldsymbol{G}}|$. Then* $\mathrm{Hom}(H_{\mathscr{Z}}, R)^\epsilon = 0$.

Proof. Since $H_{\mathscr{Z}}$ is $|\boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}}|$-torsion (as $H_{\mathscr{Z}} \subset H^1(F, \boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}})$), we have $\mathrm{Hom}(H_{\mathscr{Z}}, R) = 0$ if $p \nmid |\boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}}|$. Looking at Table 1, we see that $p \mid |\boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}}|$ if and only if either $\boldsymbol{G}$ is of type ${}^1\mathrm{A}^{(d)}_{3d-1,r}$ with $p \mid d$, or $\boldsymbol{G}$ is of type ${}^1\mathrm{E}^{16}_{6,2}$ and $p = 3$ (recall that $p \neq 2$).

(a) Assume that $G = \mathrm{SL}_3(D)$ for some central division algebra D over F with reduced degree d. Then $\boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}} \simeq (\boldsymbol{\mu}_d)^3$ and $\boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}} = \ker((\mathbf{G}_{\mathrm{m}})^3 \xrightarrow{\Pi} \mathbf{G}_{\mathrm{m}} \xrightarrow{(-)^d} \mathbf{G}_{\mathrm{m}})$, with W acting via the isomorphism $W \simeq S_3$. We let $\boldsymbol{T} = \ker((\mathbf{G}_{\mathrm{m}})^3 \xrightarrow{\Pi} \mathbf{G}_{\mathrm{m}})$ be the maximal subtorus of $\boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}}$. There is a short exact sequence of algebraic F-groups of multiplicative type

$$1 \to \boldsymbol{T} \to \boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}} \xrightarrow{\Pi} \boldsymbol{\mu}_d \to 1.$$

Since $\boldsymbol{T}$ is a torus, the product map induces an isomorphism at the level of H^1. Thus $H_{\mathscr{Z}}$ is the kernel of the map induced at the level of H^1 by the product map $\boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}} \xrightarrow{\Pi} \boldsymbol{\mu}_d$, hence

$$H_{\mathscr{Z}} \simeq \{(x, y, z) \in (F^*/F^{*d})^3 \mid xyz = 1\}$$

with W acting via the isomorphism $W \simeq S_3$. Note also that W acts trivially on $H^1(F, \boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}})/H_{\mathscr{Z}}$. There is an isomorphism

$$\mathrm{Hom}(H_{\mathscr{Z}}, R)^\epsilon \simeq \mathrm{Hom}(H_{\mathscr{Z}}/H^\epsilon_{\mathscr{Z}}, R)$$

where $H^\epsilon_{\mathscr{Z}}$ is the subgroup of $H_{\mathscr{Z}}$ generated by the elements of the form $h(s_\alpha \cdot h)$ and $h(s_\beta \cdot h)$ with $h \in H_{\mathscr{Z}}$, that is those elements of the form (x, x, x^{-2}) and (x^{-2}, x, x) with $x \in F^*/F^{*d}$. A simple computation shows that the map $H_{\mathscr{Z}} \to F^*/F^{*d}$ given by $(x, y, z) \mapsto xz^{-1}$ induces an isomorphism

$$H_{\mathscr{Z}}/H^\epsilon_{\mathscr{Z}} \xrightarrow{\sim} (F^*/F^{*d})/(F^*/F^{*d})^3.$$

Therefore, $\mathrm{Hom}(H_{\mathscr{Z}}, R)^\epsilon = 0$ if $p \neq 3$.

(b) Assume that $\boldsymbol{G}$ is of type ${}^1\mathrm{A}^{(d)}_{3d-1,2}$, i.e. $G_{\mathrm{sc}} = \mathrm{SL}_3(D)$ for some central division algebra D over F with reduced degree d. Let $\boldsymbol{\mathscr{Z}}'$ denote the inverse image of $\boldsymbol{\mathscr{Z}}$ in $\boldsymbol{G}_{\mathrm{sc}}$. Since the homomorphism $\boldsymbol{\mathscr{Z}}_{\mathrm{sc}} \to \boldsymbol{\mathscr{Z}}$ factors through $\boldsymbol{\mathscr{Z}}'$, there is an inclusion $H_{\mathscr{Z}'} \subset H_{\mathscr{Z}}$, hence an exact sequence

$$0 \to \mathrm{Hom}(H_{\mathscr{Z}}/H_{\mathscr{Z}'}, R)^{\epsilon} \to \mathrm{Hom}(H_{\mathscr{Z}}, R)^{\epsilon} \to \mathrm{Hom}(H_{\mathscr{Z}'}, R)^{\epsilon}.$$

By *(a)*, $\mathrm{Hom}(H_{\mathscr{Z}'}, R)^{\epsilon} = 0$ if $p \neq 3$ and W acts trivially on $H^1(F, \boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}})/H_{\mathscr{Z}'}$. Thus W acts trivially on the submodule $H_{\mathscr{Z}}/H_{\mathscr{Z}'}$, hence $\mathrm{Hom}(H_{\mathscr{Z}}/H_{\mathscr{Z}'}, R)^{\epsilon} = 0$ since $p \neq 2$. Therefore, $\mathrm{Hom}(H_{\mathscr{Z}}, R)^{\epsilon} = 0$ if $p \neq 3$.

(c) Assume that $\boldsymbol{G}$ is of type ${}^1\mathrm{A}^{(d)}_{3d-1,2}$ and $p = 3$. We write $d = 3^r d'$ with $3 \nmid d'$ so that $\boldsymbol{Z}_{\boldsymbol{G}_{\mathrm{sc}}} \simeq \boldsymbol{\mu}_{3^{r+1}} \times \boldsymbol{\mu}_{d'}$. Since $3 \nmid |\boldsymbol{Z}_{\boldsymbol{G}}|$ by assumption, the homomorphism $\boldsymbol{G}_{\mathrm{sc}} \to \boldsymbol{G}$ factors through $\boldsymbol{G}_{\mathrm{sc}}/\boldsymbol{\mu}_{3^{r+1}}$. Let $\boldsymbol{\mathscr{Z}}'$ (resp. $\boldsymbol{\mathscr{Z}}''$) denote the inverse image of $\boldsymbol{\mathscr{Z}}$ in $\boldsymbol{G}_{\mathrm{sc}}$ (resp. $\boldsymbol{G}_{\mathrm{sc}}/\boldsymbol{\mu}_{3^{r+1}}$). There are inclusions $H_{\mathscr{Z}'} \subset H_{\mathscr{Z}''} \subset H_{\mathscr{Z}}$, hence an exact sequence

$$0 \to \mathrm{Hom}(H_{\mathscr{Z}}/H_{\mathscr{Z}''}, R)^{\epsilon} \to \mathrm{Hom}(H_{\mathscr{Z}}, R)^{\epsilon} \to \mathrm{Hom}(H_{\mathscr{Z}''}, R)^{\epsilon}.$$

By *(a)*, W acts trivially on $H^1(F, \boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}})/H_{\mathscr{Z}'}$. Thus W acts trivially on the subquotient $H_{\mathscr{Z}}/H_{\mathscr{Z}''}$, hence $\mathrm{Hom}(H_{\mathscr{Z}}/H_{\mathscr{Z}''}, R)^{\epsilon} = 0$ since $p \neq 2$. Let $\boldsymbol{T}$ be the maximal subtorus of $\boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}'}$ as in *(a)*. There is a commutative diagram of algebraic F-groups of multiplicative type

$$\begin{tikzcd}
1 \arrow[r] & \boldsymbol{\mu}_3 \arrow[r] \arrow[d, hook] & \boldsymbol{\mu}_{3^{r+1}} \arrow[r] \arrow[d, hook] & \boldsymbol{\mu}_{3^r} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \boldsymbol{T} \arrow[r] \arrow[d, two heads, "(-)^3"] & \boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}'} \arrow[r, "\Pi"] \arrow[d, two heads] & \boldsymbol{\mu}_d \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & \boldsymbol{T} \arrow[r] & \boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}''} \arrow[r] & \boldsymbol{\mu}_{d'} \arrow[r] & 1
\end{tikzcd}$$

whose rows and columns are exact. Thus $H_{\mathscr{Z}''}$ is the kernel of the map induced at the level of H^1 by the maps $\boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}} \xrightarrow{\Pi} \boldsymbol{\mu}_d \xrightarrow{(-)^{3^r}} \boldsymbol{\mu}_{d'}$. Letting $H^{(3)}_{\mathscr{Z}''}$ denote the 3^r-torsion subgroup of $H_{\mathscr{Z}''}$, we deduce that $H^{(3)}_{\mathscr{Z}''} \simeq (F^*/F^{*3^r})^3$ with W acting via the isomorphism $W \simeq S_3$ and $H_{\mathscr{Z}''}/H^{(3)}_{\mathscr{Z}''}$ is d'-torsion. A simple computation as in *(a)* shows that $\mathrm{Hom}(H^{(3)}_{\mathscr{Z}''}, R)^{\epsilon} = 0$ (using the fact that $p \neq 2$), hence $\mathrm{Hom}(H_{\mathscr{Z}''}, R)^{\epsilon} = 0$. Therefore, $\mathrm{Hom}(H_{\mathscr{Z}}, R)^{\epsilon} = 0$.

(d) Assume that $\boldsymbol{G}$ is of type ${}^1\mathrm{E}^{16}_{6,2}$ and $p = 3$. Let α denote the relative simple root corresponding to the distinguished vertex of the Tits index of $\boldsymbol{G}$ with 3 neighbors and β denote the other relative simple root (see Table 1). Then $\boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}} \simeq \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$ and the action of s_α permutes the two copies of $\boldsymbol{\mu}_3$ whereas s_β acts trivially. Thus s_β acts trivially on $H_{\mathscr{Z}}$ so that $\mathrm{Hom}(H_{\mathscr{Z}}, R)^{\epsilon} = 0$ since $p \neq 2$. □

Remark 2.40. Keep the assumptions on $\boldsymbol{G}$ but assume that $p = 2$.

1. (1) If $\boldsymbol{G}$ is quasi-split and $p \nmid |\boldsymbol{Z}_{\boldsymbol{G}}|$, then the image of (2.12) is zero. Indeed, using Lemma 2.15(1) and proceeding as in the proof of Lemma 2.37, we see that any extension in the image of (2.12) is trivial on $Z_{\mathscr{Z}}$, and $H_{\mathscr{Z}} = 0$ since $\boldsymbol{\mathscr{Z}}_{\mathrm{sc}} = 1$. Likewise, (2.12) is zero when $G = \mathrm{SL}_3(D)$ for some division algebra D over F with reduced degree d, since in this case $Z_{\mathscr{Z}}/Z_\alpha Z_\beta \simeq F^{*d}/(F^{*d})^3$ and $\mathrm{Hom}(H_{\mathscr{Z}}, R)^{\epsilon} = 0$ (see *(a)* in the proof of Lemma 2.39).
2. (2) In any case, we deduce from the short exact sequence (2.36) and the proofs of Lemmas 2.37 and 2.39 that the image of (2.12) is killed by the highest power of 2 dividing the product of the exponents of the groups $\boldsymbol{Z}_{\boldsymbol{G}}$ and $\boldsymbol{Z}_{\boldsymbol{\mathscr{Z}}_{\mathrm{sc}}}$. Looking at Table 1, we see that this power is at most 16 (when $\boldsymbol{G}$ is of type ${}^1\mathrm{D}^{(2)}_{7,2}$).

2.7. **Conclusion.** Finally, we compute the R-module $\mathrm{Ext}^1_G(1, \mathrm{St})$.

Proposition 2.41. *Assume that $\Delta = \{\alpha\}$ and $p \neq 2$. If the adjoint action of $\mathscr{Z}$ on $\overline{U} \setminus \{1\}$ is transitive, then there is an isomorphism*

$$\mathrm{Ext}^1_G(1, \mathrm{St}) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathscr{Z}}(1,1)^{s_\alpha = -1}.$$

Proof. This follows from Propositions 2.23 and 2.29. □

Remark 2.42. Assume that $G = \mathrm{GL}_2(D) \times (D^*)^r$ for some division algebra D over F. Then the assumptions of Proposition 2.41 are satisfied when $p \neq 2$, but the result holds true when $p = 2$ (see Remark 2.30 and Lemma 2.10(2)). Letting E denote the center of D, we obtain an isomorphism

$$\mathrm{Ext}^1_G(1, \mathrm{St}) \simeq \mathrm{Hom}(E^*, R).$$

When $D = \mathbf{Q}_p$, this result is originally due to one of us (see [11, Th. VII.4.18]).

Proposition 2.43. *Assume that $|\Delta| > 1$. If $\Delta = \{\alpha, \beta\}$ with α and β non-orthogonal assume moreover that $p \neq 2$. Then $\mathrm{Ext}^1_G(1, \mathrm{St}) = 0$.*

Proof. If $|\Delta| > 2$ or $\Delta = \{\alpha, \beta\}$ with $\alpha \perp \beta$, then (2.12) is zero by Lemma 2.15(2) and the result follows from Proposition 2.23. Assume that $\Delta = \{\alpha, \beta\}$ with α and β non-orthogonal and $p \neq 2$. Using Lemmas 2.11 and 2.10(2), we can assume that $\boldsymbol{G}$ is almost-simple. Replacing F by a finite separable extension, we can assume that $\boldsymbol{G}$ is absolutely almost-simple. Using Lemma 2.11(2), we can assume that $p \nmid |\boldsymbol{Z_G}|$. Then the result follows from Propositions 2.23 and 2.31. □

Remark 2.44. Assume that $\Delta = \{\alpha, \beta\}$ with α and β non-orthogonal and $p = 2$. Proceeding as in the proof of Proposition 2.43 and using Remark 2.40(1), we see that $\mathrm{Ext}^1_G(1, \mathrm{St}) = 0$ in the following cases: $\boldsymbol{G}$ is quasi-split or G is a Levi subgroup of $\mathrm{GL}_n(D)$ for some division algebra D over F.

3. The Kottwitz set $B(G)$

This section is devoted to a brief review of basic facts concerning the theory of σ-conjugacy classes, due to Kottwitz. We fix in the sequel a finite extension F of $\mathbf{Q}_p$ and an algebraic closure $\overline{F}$ of F. Let C be the completion of $\overline{F}$, let $F^{\mathrm{nr}} \subset \overline{F}$ be the maximal unramified extension of F and finally let $\overline{\breve{F}} \simeq \breve{F} \otimes_{F^{\mathrm{nr}}} \overline{F}$ be the algebraic closure of $\breve{F}$ inside C. Let σ be the (relative to F) Frobenius automorphism on F^{nr} and $\breve{F}$. Let $\mathscr{G}_F = \mathrm{Gal}(\overline{F}/F)$, with inertia subgroup $I_F = \mathscr{G}_{F^{\mathrm{nr}}} \simeq \mathscr{G}_{\breve{F}} := \mathrm{Gal}(\overline{\breve{F}}/\breve{F})$ and Weil subgroup W_F. Finally, let G be a connected reductive group defined over F.

3.1. **Various incarnations of $B(G)$.** In [42] and [43] Kottwitz attached to G a pointed set $B(G)$. This set has now several incarnations which nicely complement each other. We will review them briefly.

3.1.1. *Via Galois cohomology.* The first definition is a cohomological one [42, 1.7], [43, 1.1-1.5]:

$$B(G) := H^1(\sigma^{\mathbf{Z}}, G(\breve{F}))$$

is the (pointed) set of *σ-equivalence classes* in $G(\breve{F})$, two elements $b, b' \in G(\breve{F})$ being in the same equivalence class if $b' = gb\sigma(g)^{-1}$ for some $g \in G(\breve{F})$, or equivalently if the elements $b\sigma, b'\sigma \in G(\breve{F}) \rtimes \langle\sigma\rangle$ are conjugate under $G(\breve{F})$. We will write $[b] = \{gb\sigma(g)^{-1} |\, g \in G(\breve{F})\} \in B(G)$ for the σ-equivalence class of $b \in G(\breve{F})$.

We have a natural isomorphism[12]

$$B(G) \xrightarrow{\sim} H^1(W_F, G(\overline{\breve{F}}))$$

induced by the inflation-restriction sequence associated to the exact sequence of topological groups $1 \to I_F \to W_F \to \sigma^{\mathbf{Z}} \to 1$ and the vanishing $H^1(\breve{F}, G) = 0$, the latter being a consequence of

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[12] Here and below, to simplify the notation, we will write $H^1(\breve{F}, G)$ etc. for the Galois cohomology $H^1(\mathscr{G}_{\breve{F}}, G(\overline{\breve{F}}))$.

Steinberg's theorem [64, Th. 1.9], since G is connected and $\breve{F}$ has cohomological dimension 1. The natural map, obtained using the restriction $W_F \to \mathscr{G}_F$ and the inclusion $G(\overline{F}) \subset G(\overline{\breve{F}})$,

$$H^1(F,G) \to H^1(W_F, G(\overline{\breve{F}})) \xleftarrow{\sim} B(G)$$

is injective.

Example 3.1 (Torus). A fundamental result of Kottwitz [42, Sec. 2] gives a functorial isomorphism $X_*(T)_{\mathscr{G}_F} \xrightarrow{\sim} B(T)$ for F-tori T, uniquely pinned down by the requirement that in the induced isomorphism $B(\mathbb{G}_m) \simeq \mathbf{Z}$ the element $1 \in \mathbf{Z}$ corresponds to the σ-conjugacy class in $\breve{F}^*$ consisting of elements with normalized valuation 1. Concretely, the isomorphism $X_*(T)_{\mathscr{G}_F} \xrightarrow{\sim} B(T)$ sends the class of $\mu \in X_*(T)$ to $[N_{E/E_0}(\mu(\pi_E))]$, where E is a finite Galois extension of F inside $\overline{F}$ splitting T, π_E is a uniformizer of E, E_0 is the maximal subfield of E unramified over F and finally $N_{E/E_0} : T(E) \to T(E_0)$ is the norm map.

3.1.2. *Via G-isocrystals.* The second incarnation (see [42, Sec. 3]) of $B(G)$ is as the set of isomorphism classes of G-*isocrystals* (relative to $\breve{F}/F$), i.e., exact, faithful F-linear tensor functors[13] $\mathrm{Rep}_F(G) \to \mathrm{Mod}_{\breve{F}}(\varphi)$, where $\mathrm{Rep}_F(G)$ is the category of finite dimensional algebraic F-representations of G and $\mathrm{Mod}_{\breve{F}}(\varphi)$ is the category of isocrystals relative to $\breve{F}/F$.

There is an equivalence (even an isomorphism) of categories between the grupoid of G-isocrystals and the grupoid having as objects the set $G(\breve{F})$, the set of morphisms $b \to b'$ being the set of $g \in G(\breve{F})$ with $gb\sigma(g)^{-1} = b'$. Specifically, every $b \in G(\breve{F})$ yields a G-isocrystal N_b:

$$N_b : \mathrm{Rep}_F(G) \to \mathrm{Mod}_{\breve{F}}(\varphi), \quad (V,\rho) \mapsto (V \otimes_F \breve{F}, b\sigma := \rho(b)(\mathrm{id}_V \otimes \sigma)),$$

whose isomorphism class depends only on $[b]$. Sending $[b]$ to the isomorphism class of N_b yields a bijection between $B(G)$ and the set of isomorphism classes of G-isocrystals. For instance, $B(\mathbb{GL}_n)$ is identified with the set of isomorphism classes of n-dimensional isocrystals [58, Rem. 3.4 (ii)].

3.1.3. *Via the Fargues-Fontaine curve.* This incarnation of $B(G)$, due to Fargues [23], will not be used in the rest of the paper, but is particularly appealing and we feel that it gives a better understanding of many of the constructions to come. We fix an embedding of the residue field $\mathbf{F}_q$ of F into the residue field $\overline{\mathbf{F}}_p$ of C and consider the Fargues-Fontaine curve $X = X_{F,\overline{\mathbf{F}}_p}$ over $\mathrm{Spec}(F)$ attached to F and $\overline{\mathbf{F}}_p$. There exists an exact, faithful, F-linear tensor functor

$$\mathscr{E} : \mathrm{Mod}_{\breve{F}}(\varphi) \to \mathrm{Bun}_X,$$

where Bun_X is the category of vector bundles on X. While *not* fully faithful, this functor is essentially surjective and induces an equivalence of categories between the isoclinic isocrystals and the semistable vector bundles on X as well as a bijection

$$\mathscr{E} : |\mathrm{Mod}_{\breve{F}}(\varphi)| \simeq |\mathrm{Bun}_X|$$

between the sets of isomorphism classes of the corresponding objects[14]. Every $b \in G(\breve{F})$ yields therefore an exact, faithful, F-linear tensor functor

$$\mathscr{E}_b = \mathscr{E} \circ N_b : \mathrm{Rep}_F(G) \to \mathrm{Bun}_X,$$

i.e., a G-bundle on X in the Tannakian sense[15] and we have the following beautiful result:

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[13] G-isocrystals can be defined for any linear algebraic group G over F. In that case one adds an assumption that the defining functor is strictly compatible with the fiber functors. If the group G is connected, as it is the case in this paper, this assumption is not necessary by the vanishing theorem of Steinberg [17, Lemma 9.1.5].

[14] Despite its innocuous-looking character, this is one of the most difficult results in the book of Fargues and Fontaine [24].

[15] There is a natural equivalence between the category of G-bundles on X and the category of G-torsors on X locally trivial for the étale topology: if Y is G-torsor étale locally trivial, we obtain a G-bundle by sending $(V,\rho) \in \mathrm{Rep}_F(G)$ to $Y \times_{G,\rho} V$; conversely, each G-bundle ω yields a locally trivial G-torsor $\mathrm{Isom}^{\otimes}(\omega_{\mathrm{can}}, \omega)$, where $\omega_{\mathrm{can}}(V,\rho) = V \otimes_F \mathscr{O}_X$.

Theorem 3.2 (Fargues, [23, Th. 5.1])**.** *The construction $b \to \mathscr{E}_b$ yields bijections of pointed sets*

$$B(G) \simeq |\mathrm{Bun}_G| \simeq H^1_{\text{ét}}(X, G),$$

where $|\mathrm{Bun}_G|$ is the set of isomorphism classes of G-bundles on X.

3.2. **The structure of $B(G)$.** In order to describe $B(G)$ Kottwitz defined two rather technical but very important maps, which we briefly review.

3.2.1. *The Newton map.* Let $b \in G(\breve{F})$ and let N_b be the associated G-isocrystal. If $\mathbb{D}$ is the pro-torus over F with character group $\mathbf{Q}$ (the "slope torus") there is a unique morphism

$$\nu_b : \mathbb{D}_{\breve{F}} \to G_{\breve{F}}$$

such that for all $(V, \rho) \in \mathrm{Rep}_F(G)$ the composition $\rho \circ \nu_b : \mathbb{D}_{\breve{F}} \to \mathrm{GL}(V \otimes_F \breve{F})$ corresponds to the (Dieudonné-Manin) slope decomposition of $N_b(V)$, considered as a $\mathbf{Q}$-grading on $V \otimes_F \breve{F}$. The homomorphism ν_b is the *Newton or slope map attached to b* [42, Sec. 4]. It satisfies

$$\nu_{\sigma(b)} = \sigma(\nu_b),\ \nu_b = \mathrm{Int}(b) \circ \sigma(\nu_b),\ \nu_{gb\sigma(g)^{-1}} = \mathrm{Int}(g) \circ \nu_b = g\nu_b g^{-1},$$

hence the $G(\breve{F})$-conjugacy class $[\nu_b]$ of ν_b depends only on $[b] \in B(G)$ and is σ-invariant, thus

$$[\nu_b] \in \mathscr{N}(G) := [\mathrm{Hom}_{\breve{F}}(\mathbb{D}_{\breve{F}}, G_{\breve{F}})/\mathrm{Int}\, G(\breve{F})]^{\sigma=1}.$$

The elements of $\mathscr{N}(G)$ are called *Newton vectors.*

Remark 3.3. By choosing a maximal torus T of G defined over F, we obtain a more concrete description $\mathscr{N}(G) \simeq (X_*(T)_{\mathbf{Q}}/W)^{\mathscr{G}_F}$, where W is the absolute Weyl group of G with respect to T. If we also choose a basis Δ for the root system, each element of $\mathscr{N}(G)$ has a unique representative in $X_*(T)^+_{\mathbf{Q}} := \{\nu \in X_*(T)_{\mathbf{Q}} |\, \langle \nu, \alpha \rangle \geq 0\, \forall\, \alpha \in \Delta\}$. One infers from this a partial order on $\mathscr{N}(G)$, induced from the natural partial order on $X_*(T)^+_{\mathbf{Q}}$ (for which $\nu_1 \leq \nu_2$ if $\nu_2 - \nu_1 \in \sum_{\alpha \in \Delta} \mathbf{Q}_{\geq 0}\ \alpha^\vee$). Pulling back along the Newton map we deduce a partial order on $B(G)$.

Sending $[b] \in B(G)$ to $[\nu_b] \in \mathscr{N}(G)$ yields the *Newton map*

$$\nu : B(G) \to \mathscr{N}(G).$$

Kottwitz proves [42, 4.5] that ν_b is trivial if and only if $[b] \in H^1(F, G) \subset B(G)$.

Example 3.4. If $G = T$ is an F-torus then $\mathscr{N}(T) \simeq \mathrm{Hom}_F(\mathbb{D}, T) \simeq X_*(T)^{\mathscr{G}_F}_{\mathbf{Q}}$ and ν_b (for $b \in T(\breve{F})$) is the unique element of $X_*(T)^{\mathscr{G}_F}_{\mathbf{Q}}$ such that $\langle \lambda, \nu_b \rangle = v_F(\lambda(b))$ for $\lambda \in X^*(T)^{\mathscr{G}_F}$, where v_F is the normalized valuation on $\breve{F}$. The composite map[16]

$$X_*(T) \to X_*(T)_{\mathscr{G}_F} \xrightarrow{\sim} B(T) \xrightarrow{\nu} X_*(T)^{\mathscr{G}_F}_{\mathbf{Q}}$$

sends $\mu \in X_*(T)$ to its Galois average $\mu^\diamond$. We have an exact sequence $0 \to H^1(F, G) \to B(T) \xrightarrow{\nu} X_*(T)^{\mathscr{G}_F}_{\mathbf{Q}}$.

Example 3.5. For $G = \mathbb{GL}_n$ the partially ordered set $\mathscr{N}(G)$ is naturally in bijection with

$$(\mathbb{Q}^n)_+ := \{(x_1, \ldots, x_n) \in \mathbb{Q}^n |\, x_1 \geq \cdots \geq x_n\},$$

endowed with the usual dominance order, for which $(y_1, \ldots, y_n) \leq (x_1, \ldots, x_n)$ if and only if $x_1 + \cdots + x_i \geq y_1 + \cdots + y_i$ for all i, with equality for $i = n$. An element $x = (x_1^{(n_1)}, \ldots, x_r^{(n_r)}) \in \mathscr{N}(G)$ is thus given by rational numbers $x_1 > x_2 > \cdots > x_r$ and multiplicities $n_1, \ldots, n_r \in \mathbf{Z}_{>0}$ such that $\sum_{i=1}^r n_i = n$. If $[b] \in B(G)$ and $x_1, \ldots, x_r$ are the slopes of the isocrystal N_b (in decreasing order), then $\nu([b]) = (x_1^{(n_1)}, \ldots, x_r^{(n_r)})$, where n_i are the dimensions of the isotypic parts of N_b. The Newton map ν is injective with image consisting of those $x = (x_1^{(n_1)}, \ldots, x_r^{(n_r)}) \in (\mathbb{Q}^n)_+$ such that $n_i x_i \in \mathbf{Z}, i = 1, \ldots, r$ (see [58, Ex. 1.19]).

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[16] Cf. example 3.1 for the isomorphism in the displayed formula.

3.2.2. *The Kottwitz map κ.* Let $\pi_1(G)$ be the (Borovoi) algebraic fundamental group of G [7], [58, 1.13]. It is a finitely generated discrete $\mathscr{G}_F$-module, functorial and exact in G, and isomorphic (as abelian group) to $X_*(T)/\sum_{\alpha\in\Phi(G,T)}\mathbf{Z}\alpha^\vee$, where T is a maximal torus of $G_{\overline{F}}$ and $\Phi(G,T)$ is the set of roots of T, $\alpha^\vee$ being the co-root corresponding to $\alpha\in\Phi(G,T)$. For example, $\pi_1(\mathbb{GL}_n)\simeq\mathbf{Z}$ and $\pi_1(T)\simeq X_*(T)$ for an F-torus T. We also note [58, 1.14] that $\pi_1(G)\simeq\pi_1(G')$ for any inner form G' of G, and that $\pi_1(G)\xrightarrow{\sim}\pi_1(G_{\mathrm{ab}}:=G/G_{\mathrm{der}})$ whenever G_{der} is simply connected.

By a theorem of Kottwitz (see [43, Sec. 6]), [58, Th. 1.15]) there is a unique natural map[17]

$$\kappa: B(G)\to\pi_1(G)_{\mathscr{G}_F}$$

making the following diagram commute:

$$\begin{tikzcd}
\breve{F}^* \arrow[r, "v_F"] \arrow[d] & \mathbf{Z} \arrow[d, equal] \\
B(\mathbb{G}_m) \arrow[r, "\kappa"] & \pi_1(\mathbb{G}_m)_{\mathscr{G}_F}.
\end{tikzcd}$$

For instance for $G=\mathbb{GL}_n$ the map κ sends $[b]\in B(G)$ to $v_F(\det b)$. In general, the induced map $(\nu,\kappa): B(G)\to\mathscr{N}(G)\times\pi_1(G)_{\mathscr{G}_F}$ is injective and there exists a natural map of exact sequences

$$\begin{tikzcd}
H^1(F,G) \arrow[r, hook] \arrow[d, "\kappa", "\wr"'] & B(G) \arrow[r, "\nu"] \arrow[d, "\kappa"] & \mathscr{N}(G) \arrow[d, "\delta"] \\
\pi_1(G)_{\mathscr{G}_F,\mathrm{tors}} \arrow[r, hook] & \pi_1(G)_{\mathscr{G}_F} \arrow[r, "\mu\mapsto\mu^\diamond"] & \pi_1(G)_{\mathbf{Q}}^{\mathscr{G}_F}
\end{tikzcd}$$

Fargues [23, Sec. 8], inspired by Labesse's [44] reinterpretation of the constructions of Kottwitz, also gave a geometric interpretation of the Kottwitz map κ using the Fargues-Fontaine curve X:

(1) Using cohomology theory of crossed modules [44], he defined a map $\kappa^F: H^1_{\text{ét}}(X,G)\to\pi_1(G)_{\mathscr{G}_F}$ which agrees with the Kottwitz map κ after the identification $B(G)\xrightarrow{\sim}H^1_{\text{ét}}(X,G)$.
(2) Then, using the universal G-torsor and cohomology of cross modules again, he defined a Chern class map $c_1^G: H^1_{\text{ét}}(X,G)\to\widehat{\pi_1(G)}_{\mathscr{G}_F}$ (the profinite completion) and showed that
$$\kappa^F(b)=-c_1^G(\mathscr{E}_b),\quad b\in G(\breve{F}).$$

3.2.3. *Automorphism groups.* Let $b\in G(\breve{F})$ and let J_b be the automorphism group of the G-isocrystal N_b, i.e., the connected reductive group over F such that for any F-algebra R

$$J_b(R):=\{g\in G(\breve{F}\otimes_F R)|\,gb\sigma(g)^{-1}=b\}.$$

Clearly, if $b=1$ then $J_b=G$. Suppose that G is quasi-split and let M_b be the centralizer of ν_b, a Levi component of an F-parabolic of G. By results of Kottwitz [42, Sec. 6] the group J_b is an inner form of M_b.

3.2.4. *Basic and decent elements.* The subset $B(G)_{\mathrm{basic}}$ of *basic elements of* $B(G)$ consists of those $[b]\in B(G)$ for which ν_b factors through $Z_{\breve{F}}$, where Z is the center of G. If $[b]\in B(G)_{\mathrm{basic}}$ the Newton map $\nu_b:\mathbb{D}_{\breve{F}}\to Z_{\breve{F}}$ is defined over F, since its conjugacy class is σ-invariant. It follows from results of Kottwitz (see [42] and [58]) that an element $[b]\in B(G)$ is in $B(G)_{\mathrm{basic}}$ if and only if it satisfies one of the following equivalent conditions:

- The automorphism group J_b is an inner form of G.
- $[b]$ is a minimal element for the partial order on $B(G)$.

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[17] Kottwitz formulated his theorem in [42], [43] in terms of the center $Z(\widehat{G})$ of the Langlands dual group. The formulation we present here in terms of the algebraic fundamental group is due to Rapoport-Richartz [58]. It has better functoriality properties than the original one.

• The associated G-bundle $\mathscr{E}_b$ on the Fargues-Fontaine curve X is semistable (i.e., $\mathscr{E}_b(\mathrm{Lie}(G),\mathrm{Ad})$ is a semi-stable vector bundle on X, or equivalently[18] $\mathscr{E}_b(V,\rho)$ is semi-stable for any homogeneous representation ρ of G). For $G=\mathbb{GL}_n$ the set $B(G)_{\mathrm{basic}}$ corresponds to that of isoclinic isocrystals of dimension n.

Kottwitz [42, ch. 5] proved that the Kottwitz map κ induces a bijection

$$\kappa : B(G)_{\mathrm{basic}} \simeq \pi_1(G)_{\mathscr{G}_F}.$$

Since κ identifies $H^1(F,G)$ and $\pi_1(G)_{\mathscr{G}_F,\mathrm{tors}}$, it follows that $H^1(F,G)\subset B(G)_{\mathrm{basic}}$. It also follows that when G_{der} is simply connected the map $G\to G_{\mathrm{ab}}:=G/G_{\mathrm{der}}$ induces a bijection $B(G)_{\mathrm{basic}}\xrightarrow{\sim} B(G_{\mathrm{ab}})$. In particular, $B(G)_{\mathrm{basic}}$ is trivial if G is semisimple and simply connected.

Let now $F=\mathbf{Q}_p$ and let s be a positive integer. We say that $b\in G(\breve{\mathbf{Q}}_p)$ is *s-decent* if $s\nu_b$ factors through the quotient $\mathbb{G}_{m\breve{\mathbf{Q}}_p}$ of $\mathbb{D}_{\breve{\mathbf{Q}}_p}$ and we have an equality in $G(\breve{\mathbf{Q}}_p)\rtimes\sigma^{\mathbf{Z}}$:

$$(b\sigma)^s=(s\nu_b)(p)\sigma^s.$$

By [60, Cor. 1.9] this implies that ν_b is defined over $\mathbf{Q}_{p^s}$ and $b\in G(\mathbf{Q}_{p^s})$. We say that $[b]\in B(G)$ is *decent* if it contains an s-decent element for some positive integer s. Since G is connected and the residue field of $\breve{\mathbf{Q}}_p$ is algebraically closed [17, Lemma 9.1.33] implies that any $[b]\in B(G)$ is decent (see also [42]). Moreover, by [17, Lemma 9.6.19], if G is quasi-split each $[b]\in B(G)$ contains an element b which is s-decent for some s and such that ν_b is defined over $\mathbf{Q}_p$.

4. PERIOD DOMAINS

The period domains we are interested in classify weakly admissible filtrations on isocrystals. We briefly review here the relevant facts. A beautiful reference for everything that follows is [17, Ch. I, IV, V, VI, VIII].

4.1. **Filtrations.** We start by reviewing some basic facts concerning filtrations of Tannaka fiber functors, following [17].

4.1.1. *Filtrations on vector spaces.* If K is a field we denote by Vec_K the category of finite dimensional K-vector spaces. An $\mathbf{R}$-filtration FV on $V\in\mathrm{Vec}_K$ is a decreasing, exhaustive, separated filtration $(F^xV)_{x\in\mathbf{R}}$ by K-vector subspaces of V, such that $F^xV_K=\cap_{y<x}F^yV_K$ for all x. We denote by $\mathrm{gr}^x(FV)=F^xV/\sum_{y>x}F^yV$ and $\mathrm{gr}(FV)=\oplus_{x\in\mathbf{R}}\mathrm{gr}^x(FV)$ the associated $\mathbf{R}$-graded vector space. The *type of* FV is the nonincreasing sequence $\nu(FV)=(x_1^{(n_1)},\dots,x_r^{(n_r)})\in\mathbf{R}^n$, where $x_1>\cdots>x_r$ are the jumps of the filtration (i.e., those x for which $\mathrm{gr}^x(FV)\neq 0$) and x_i is repeated $n_i:=\dim\mathrm{gr}^{x_i}(FV)$ times. We say that FV is a *$\mathbf{Q}$-filtration* if $\nu(FV)\in\mathbf{Q}^n$.

Let K/k be a field extension and let Fil_k^K be the category of pairs (V,FV_K), where $V\in\mathrm{Vec}_k$ and FV_K is an $\mathbf{R}$-filtration on the K-vector space $V_K:=V\otimes_k K$. This is a quasi-abelian category endowed with the degree, rank and slope functions, defined, for $V:=(V,FV_K)$, by

$$\deg(V)=\sum_x x\dim_K\mathrm{gr}^x(V_K),\ \mathrm{rk}(V)=\dim_k V,\ \mu(V)=\frac{\deg(V)}{\mathrm{rk}(V)},\ V\neq 0.$$

One has a good Harder-Narasimhan formalism with respect to this slope function. In particular, a notion of semi-stability for objects of Fil_k^K and the tensor product of semistable objects is semistable when K/k is separable, thanks to the theorems of Faltings [21] and Totaro [66] (this fails when K/k is not separable). Moreover, one can characterize the semistable objects in terms of the inner product (this will be a recurrent theme in the sequel) as follows. Recall that, if $FV,F'V$ are $\mathbf{R}$-filtrations on $V\in\mathrm{Vec}_K$, their *inner product* is defined by

$$\langle FV,F'V\rangle:=\sum_{x,y\in\mathbf{R}}xy\dim_K\mathrm{gr}^x_{FV}(\mathrm{gr}^y_{F'V}),$$

---

[18] This uses the deep fact that semi-stable vector bundles on X are stable under tensor product.

where $\mathrm{gr}^x_{FV}(\mathrm{gr}^y_{F'V})$ is the xth graded piece of the K-vector space $\mathrm{gr}^y(F'V)$ endowed with the filtration induced by FV. The semistability criterion is then:

Proposition 4.1 ([17, Cor. 1.2.6])**.** *The pair (V, FV_K) is semistable if and only if*

$$\langle FV_K, F'V_K\rangle \leq \mu(V, FV_K)\deg(V, F'V)$$

for all $\mathbf{Z}$-filtrations $F'V$ of V. Moreover, it suffices to check this inequality when $\deg(V, F'V) = 0$.

Let $\mathbf{Q}-\mathrm{Fil}^K_k$ be the full subcategory of those (V, FV_K) for which FV_K is a $\mathbf{Q}$-filtration on V_K and let $\mathbf{Q}-\mathrm{Grad}_K$ be the category of $\mathbf{Q}$-graded finite dimensional K-vector spaces. There are natural functors, the first being $(V, FV) \to \mathrm{gr}(FV)$ and the second being the forgetful functor

$$\mathbf{Q}-\mathrm{Fil}^K_k \to \mathbf{Q}-\mathrm{Grad}_K \to \mathrm{Vec}_K,$$

as well as a functor $\mathbf{Q}-\mathrm{Grad}_K \to \mathbf{Q}-\mathrm{Fil}^K_k$ sending $V = \oplus_{a\in\mathbf{Q}}V_a$ to the filtration FV such that $F^xV = \sum_{a\geq x} V_a$.

4.1.2. *Filtrations on* $\mathrm{Rep}_k(G)$. Let K/k be a field extension and let G be a linear algebraic group over k. Let $\omega^G : \mathrm{Rep}_k(G) \to \mathrm{Vec}_k$ be the standard fiber functor, sending (V, ρ) to V. Recall that $\mathbb{D}_K$ is the pro-torus over K with character group $\mathbf{Q}$. The functor sending $(V, \rho) \in \mathrm{Rep}_K(\mathbb{D}_K)$ to $V = \oplus_{a\in\mathbf{Q}}V_a$, where V_a is the weight space of V corresponding to $a \in \mathbf{Q} = X^*(\mathbb{D}_K)$, induces an equivalence of neutral Tannakian categories over K

$$\mathrm{Rep}_K(\mathbb{D}_K) \simeq \mathbf{Q}-\mathrm{Grad}_K.$$

Any morphism of K-group schemes $\mathbb{D}_K \to G_K$ induces therefore a $\mathbf{Q}$-*grading of* ω^G *over* K, i.e., a tensor functor $F : \mathrm{Rep}_k(G) \to \mathbf{Q}-\mathrm{Grad}_K$ whose composition with the forgetful functor $\mathbf{Q}-\mathrm{Grad}_K \to \mathrm{Vec}_K$ is $(V, \rho) \to V \otimes_k K$. All $\mathbf{Q}$-gradings of ω^G over K are thus obtained, thanks to the Tannakian formalism. If $G = \mathbb{GL}(V)$ with $V \in \mathrm{Vec}_k$, giving F comes down to giving a $\mathbf{Q}$-grading of $V \otimes_k K$.

A $\mathbf{Q}$-*filtration of* ω^G *over* K is a tensor functor $F : \mathrm{Rep}_k(G) \to \mathbf{Q}-\mathrm{Fil}^K_k$ whose composition with $\mathbf{Q}-\mathrm{Fil}^K_k \to \mathbf{Q}-\mathrm{Grad}_K$ is exact[19] and whose composition with the forgetful functor $\mathbf{Q}-\mathrm{Fil}^K_k \to \mathrm{Vec}_k$ is ω^G. When $G = \mathbb{GL}(V)$ with $V \in \mathrm{Vec}_k$, giving F comes down to giving a $\mathbf{Q}$-filtration of $V \otimes_k K$ [17, Rem. 4.2.11].

Let $\mathrm{Grad}_K(\omega^G)$ (resp. $\mathrm{Fil}_K(\omega^G)$) be the set of $\mathbf{Q}$-gradings (reps. $\mathbf{Q}$-filtrations) of ω^G over K. Composition with the natural functor $\mathbf{Q}-\mathrm{Grad}_K \to \mathbf{Q}-\mathrm{Fil}^K_k$ yields a natural map

$$\mathrm{Hom}_K(\mathbb{D}_K, G_K) \simeq \mathrm{Grad}_K(\omega^G) \to \mathrm{Fil}_K(\omega^G),$$

which is surjective when G is reductive or k has characteristic 0 (see [17, Th. 4.2.13]).

We assume in the sequel that G is reductive. Two morphisms $\lambda, \lambda' : \mathbb{D}_K \to G_K$ are in the same fiber of the map $\mathrm{Hom}_K(\mathbb{D}_K, G_K) \to \mathrm{Fil}_K(\omega^G)$ (in which case we say that they are *par-equivalent*) if and only if $\lambda' = \mathrm{Int}(g)\lambda$ for some $g \in P(\lambda)(K)$, where $P(\lambda)$ is the parabolic subgroup of G defined over K consisting of those $g \in G$ for which $\lim_{t\to 0}\mathrm{Int}(\lambda(t))g$ exists. If $U(\lambda)$ is the unipotent radical of $P(\lambda)$, λ and λ' are par-equivalent if and only if there is a unique $g \in U(\lambda)(K)$ such that $\lambda' = \mathrm{Int}(g)\lambda$, and then $P(\lambda') = P(\lambda)$ (however the latter equality does not imply that λ, λ' are par-equivalent). To summarize, we obtain maps

$$\mathrm{Hom}_K(\mathbb{D}_K, G_K)/\mathrm{par}-\mathrm{equivalence} \simeq \mathrm{Fil}_K(\omega^G) \twoheadrightarrow K-\mathrm{Par}(G),$$

where $K-\mathrm{Par}(G)$ is the set of parabolic subgroups of G defined over K.

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[19] The exactness condition is imposed so that filtrations can be described using gradings.

4.1.3. *Filtrations on isocrystals.* Consider the setup described in the introduction to section 3: F is a finite extension of $\mathbf{Q}_p$, etc. Let K be an extension of $\breve{F}$. An object of the category of filtered isocrystals over K (relative to $\breve{F}/F$) consists of an isocrystal (V,φ) over $\breve{F}$ together with a $\mathbf{Q}$-filtration FV_K on V_K. This category has a good Harder-Narasimhan formalism for the slope function

$$\mu(V,\varphi,FV_K)=\frac{1}{\dim V}\Big(\sum_{i\in\mathbf{Z}} i\dim_K \mathrm{gr}^i_F(V_K)-v_F(\det\varphi)\Big),$$

and its semistable objects of slope 0 are the weakly admissible filtered isocrystals introduced by Fontaine, and they are stable under tensor products by the theorem of Faltings and Totaro.

The above slope function can also be described as

$$\mu(V,\varphi,FV_K)=\mu(V,FV_K)+\mu(V,F_0V),$$

where F_0V is the slope filtration (defined by $F_0^xV=\bigoplus_{\alpha\leq -x}V_\alpha$) associated to the slope decomposition $(V,\varphi)=\bigoplus_{\alpha\in\mathbf{Q}}V_\alpha$ of the isocrystal (V,φ).

4.2. **Period domains.** We will define now the period domains we will be working with. Let the notation be as at the beginning of section 3.

4.2.1. *Flag varieties and period domains.* Consider a triple $(G,b,\{\mu\})$, where G is a connected reductive group over F, $b\in G(\breve{F})$, and $\{\mu\}$ is a conjugacy class of cocharacters of G over $\overline{F}$. Let $E=E(G,\{\mu\})\subset\overline{F}$ be the associated reflex field[20]. Let $\mathscr{F}=\mathscr{F}(G,\{\mu\})$ be the associated flag variety, a smooth projective variety over E, homogeneous under G_E and whose $\overline{E}=\overline{F}$-points are the par-equivalence classes of elements in $\{\mu\}$. If G is quasi-split, which will be the case in our applications, a result of Kottwitz [43, Lemma 1.1.3] shows that $\{\mu\}$ contains elements μ defined over E, and then $\mathscr{F}=G_E/P(\mu)$ (recall that $P(\mu)$ is the associated parabolic subgroup of G_E).

Let μ' be an element of $\{\mu\}$ defined over an extension K of E. We say that the *pair* (b,μ') *is weakly admissible* if the filtered isocrystal $(N_b(V),F_{\rho\mu'}(V_K))$ is weakly admissible for all $(V,\rho)\in\mathrm{Rep}_F(G)$. By the tensor product theorem it suffices to check this for a single faithful representation (V,ρ).

Let $\breve{E}=E\breve{F}$ and let $\breve{\mathscr{F}}$ be the adic space attached to $\mathscr{F}\otimes_E\breve{E}$. By [60, Prop. 1.36] there is a partially proper open subset $\breve{\mathscr{F}}^{\rm wa}=\breve{\mathscr{F}}(G,b,\{\mu\})^{\rm wa}$ of $\breve{\mathscr{F}}$ such that $\breve{\mathscr{F}}^{\rm wa}(K)$ is the set of weakly admissible points in $x\in\breve{\mathscr{F}}(K)$, i.e., points with associated cocharacter $\mu_x\in\{\mu\}$ defined over K and for which the pair (b,μ_x) is weakly admissible. This is the *period domain attached to* $(G,b,\{\mu\})$. We will see in the next section, and this will crucial for the computation of its étale cohomology, that the period domain is of the form

$$\breve{\mathscr{F}}^{\rm wa}=\breve{\mathscr{F}}\setminus\bigcup_{i\in I}J_b(F)Z_i,$$

where $\{Z_i\}_{i\in I}$ is an *explicit* finite set of Schubert varieties.

Up to isomorphism, $\breve{\mathscr{F}}(G,b,\{\mu\})^{\rm wa}$ depends only on $[b]\in B(G)$ (sending μ to $\mathrm{Int}(g)(\mu)$ yields an isomorphism $\breve{\mathscr{F}}^{\rm wa}_b\simeq\breve{\mathscr{F}}^{\rm wa}_{gb\sigma(g)^{-1}}$). The group $J_b(F)$ acts naturally on the flag variety $\breve{\mathscr{F}}$ over $\breve{E}$ (via $J_b(F)\subset G(\breve{F})$) and this action restricts to an action on $\breve{\mathscr{F}}^{\rm wa}$. Moreover, if $F=\mathbf{Q}_p$ and b is s-decent, the period domain has a canonical model $\mathscr{F}^{\rm wa}\subset\mathscr{F}\otimes_E E_s$ over $E_s:=E\mathbf{Q}_{p^s}$.

Example 4.2. Take $G=\mathbb{GL}_n$, $[b]=[1]$, and $\{\mu\}=(n-1,-1,-1,\ldots,-1)$. The corresponding period domain is $\mathbb{P}^{n-1}_F\setminus\cup_{H\in\mathscr{H}}H$, where $\mathscr{H}$ is the set of F-rational hyperplanes, i.e., the Drinfeld symmetric space of dimension $n-1$.

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[20] Recall that E is simply the field of definition of $\{\mu\}$, a finite extension of F.

4.2.2. *Existence of weakly admissible filtrations.* Fontaine and Rapoport in [25, Th. 3] found a simple criterion for the existence of weakly admissible filtrations on isocrystals; this result was extended in [17, Th. 9.5.10] (see also [59, Prop. 3.1]). To present it, we assume, for simplicity, that G is quasi-split (see [59, Sec. 3] for the general case) and we recall that the set of acceptable elements for $\{\mu\}$ [59, 2.2] is

$$A(G,\{\mu\}) := \{[b] \in B(G) | \nu_b \leq \mu^{\diamond}\},$$

a finite nonempty set, intersecting nontrivially with $B(G)_{\text{basic}}$ (see [59, Lemma. 2.5]).

The following result is a generalization (and a group-theoretical reformulation) of Mazur's "the Hodge polygon lies above the Newton polygon" property of isocrystals.

Theorem 4.3 (Fontaine-Rapoport, [17, Th. 9.5.10])**.** *The space* $\breve{\mathscr{F}}(G,b,\{\mu\})^{\text{wa}}$ *is nonempty if and only if* $[b] \in A(G,\{\mu\})$.

4.2.3. *Local Shtuka datum.* The following notion will be useful in the rest of the paper. It is a modification of the notion of a local Shimura datum of Rapoport-Viehmann [59, Def. 5.1], where we drop the assumption that μ is minuscule and allow $[b]$ to be just acceptable (instead of neutral acceptable).

Definition 4.4. A *local Shtuka datum* over F is a triple $(G,[b],\{\mu\})$ consisting of a connected reductive algebraic group G over F, a σ-conjugacy class $[b] \in B(G)$, and a geometric conjugacy class $\{\mu\}$ of cocharacters of G defined over $\overline{F}$. We assume that $[b] \in A(G,\{\mu\})$.

As discussed above, associated to a local Shtuka datum are the following data:

1. the algebraic group $J = J_b$ over F, for $b \in [b]$,
2. the reflex field $E = E(G,\{\mu\})$,
3. the flag variety $\mathscr{F} = \mathscr{F}(G,\{\mu\})$ over E,
4. the period domain $\breve{\mathscr{F}}^{\text{wa}} = \breve{\mathscr{F}}(G,b,\{\mu\})^{\text{wa}}$ over $\breve{E}$.

Remark 4.5. As explained in [17, Sec. IX.8], assuming $F = \mathbf{Q}_p$ is not really a restriction since Weil descent allows one to pass from a general F to $\mathbf{Q}_p$ (this is similar to the situation for Shimura varieties, where restriction of scalars allows one to reduce the study to groups defined over $\mathbf{Q}$). In particular, we can get the Drinfeld symmetric space over general F as a period domain in this context as well (see [17, Example 9.8.9]). *For simplicity, we will work from now on with* $F = \mathbf{Q}_p$.

5. THE GEOMETRY OF COMPLEMENTS OF PERIOD DOMAINS

Via the embeddings of period domains into flag varieties, we will reduce the computation of the cohomology of a period domain to that of its complement in the flag variety. The period domain is a locus of semistability and its complement can be stratified by Schubert varieties given by the degree to which this semistability fails[21]. This section describes this stratification.

5.1. **The Hilbert-Mumford criterion.** Suppose that a connected reductive group G over a field k acts on a proper algebraic variety X over k. The Hilbert-Mumford criterion [48] describes the semistable points of this action via eigenvalues of 1-parameter subgroups (1-PSs, for short). We will review it briefly.

Let $\mathscr{L} \in \operatorname{Pic}^G(X)$ be a G-equivariant line bundle on X. If $\lambda \in X_*(G)$ is defined over k, Mumford defined the *slope* of $x \in X(k)$ with respect to λ and $\mathscr{L}$, denoted $\mu^{\mathscr{L}}(x,\lambda) \in \mathbf{Z}$. If $x_0 = \lim_{t\to 0}\lambda(t)x$ (this limit exists since, by properness of X, the map $\mathbb{G}_m \to X, t \to \lambda(t)x$ extends to $\mathbb{A}^1$), then $\mu := \mu^{\mathscr{L}}(x,\lambda)$ is characterized by the fact that λ acts by the character $t \to t^{-\mu}$ on the fiber $\mathscr{L}_{x_0}$ (note that x_0 is fixed under the action of $\mathbb{G}_m$, so this makes sense).

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[21] This stratification shares many properties with the Harder-Narasimhan stratification of the space of vector bundles over a Riemann surface.

Since

$$\mu^{\mathscr{L}_1\otimes\mathscr{L}_2}(x,\lambda^r)=r(\mu^{\mathscr{L}_1}(x,\lambda)+\mu^{\mathscr{L}_2}(x,\lambda))$$

for $r\in\mathbf{Z}_{>0}$, the previous definition extends to $\mathscr{L}\in\mathrm{Pic}^G(X)_{\mathbf{Q}}$ and $\lambda\in X_*(G)_{\mathbf{Q}}$. Moreover, we have $\mu^{\mathscr{L}}(gx,\lambda)=\mu^{\mathscr{L}}(x,g^{-1}\lambda g)$ for $g\in G(k)$ and, most importantly [48, Prop. 2.7]

$$\mu^{\mathscr{L}}(\alpha x,\lambda)=\mu^{\mathscr{L}}(x,\lambda),\quad \alpha\in P(\lambda)(k).$$

The construction has good functoriality properties: we have $\mu^{f^*(\mathscr{M})}(x,\lambda)=\mu^{\mathscr{M}}(f(x),\lambda)$ for a G-morphism $f:X\to Y$ of G-varieties and $\mathscr{M}\in\mathrm{Pic}^G(Y)_{\mathbf{Q}}$. Moreover, if $X=X_1\times\cdots\times X_m$ is a product of G-varieties and $\mathscr{L}_i\in\mathrm{Pic}^G(X_i)_{\mathbf{Q}}$ then

$$\mu^{\mathscr{L}_1\boxtimes\cdots\boxtimes\mathscr{L}_m}((x_1,\dots,x_m),\lambda)=\sum_{i=1}^m\mu^{\mathscr{L}_i}(x_i,\lambda).$$

Let now k be algebraically closed. Recall that the (open) semistable locus $X^{\mathrm{ss}}(\mathscr{L})$ in X can be defined as the set of points $x\in X(k)$ having an affine open neighborhood of the form $X_f=\{f\neq 0\}$ with $f\in\Gamma(X,\mathscr{L}^{\otimes n})^G$ for some n. We have the following Hilbert-Mumford numerical criterion:

Theorem 5.1 (Hilbert-Mumford). *If k is algebraically closed and $\mathscr{L}$ is ample then, for $x\in X(k)$,*

$$x\in X^{\mathrm{ss}}(\mathscr{L})\iff\mu^{\mathscr{L}}(x,\lambda)\geq 0\ \textit{for all}\ \lambda\in X_*(G).$$

5.2. **A Hilbert-Mumford criterion for weak admissibility.** Since weak admissibility was defined as a semistability condition in Section 4.1.3 we can test it using a Hilbert-Mumford criterion once the right linearization of the group action is defined. We will now describe it. For the rest of this section G will be a reductive group over a perfect field k.

5.2.1. *Invariant inner products.* An *invariant inner product on G* (IIP for short) is the data of a positive-definite bilinear form on $X_*(T)_{\mathbf{Q}}$ for each maximal torus T of G (defined over $\bar{k}$), compatible[22] with the action of $G(\bar{k})$ and $\mathscr{G}_k$. These objects are standard in GIT and they seem to go back at least to Kempf's celebrated paper [39].

One can describe IIP's on G in terms of a fixed maximal torus T_0 of G (defined over $\bar{k}$) as follows. Let $W=W(G,T_0)$ be the associated Weyl group and consider the *L-action*[23] of $\mathscr{G}_k$ on $X_*(T_0)$ (if G is quasi-split over k and T_0 is defined over k, this is the usual action of $\mathscr{G}_k$ on $X_*(T_0)$). The conjugacy of maximal tori in G implies that IIP's on G correspond to $\mathscr{G}_k\ltimes W$-invariant inner products on $X_*(T_0)_{\mathbf{Q}}$, hence any G has an IIP (the action of $\mathscr{G}_k$ factors through a finite quotient).

When G is semisimple there is a natural choice of an IIP on G corresponding to the inner product on $X_*(T_0)_{\mathbf{Q}}$ given by the Killing form:

$$\mathscr{P}(\lambda,\lambda')=\sum_{\chi}\langle\lambda,\chi\rangle\langle\lambda',\chi\rangle,$$

the sum being taken over the roots of T_0. In general, [17, 6.2.4] shows that any IIP on G is the orthogonal direct sum of IIP's on the torus $Z(G)^0$ and on G_{der}. Moreover, on a k-simple semisimple group any IIP is a positive multiple of the Killing form; on the other hand, any $\mathscr{G}_k$-invariant inner product on $X_*(T)_{\mathbf{Q}}$ is an IIP on the torus T, so there is no canonical choice in this case.

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[22] That is, such that the maps $\mathrm{Int}(g):X_*(T)_{\mathbf{Q}}\to X_*(gTg^{-1})_{\mathbf{Q}}$ and $\tau:X_*(T)_{\mathbf{Q}}\to X_*({}^{\tau}T)_{\mathbf{Q}}$, ${}^{\tau}T=\tau T\tau^{-1}$, induced by any $g\in G(\bar{k})$ and $\tau\in\mathscr{G}_k$ are isometries.

[23] Explicitly, pick a Borel subgroup B_0 of G defined over $\bar{k}$ and containing T_0; if $\tau\in\mathscr{G}_k$, one can find $g\in G(\bar{k})$, unique up to left translation by $T_0(\bar{k})$, such that $g\,{}^{\tau}T_0g^{-1}=T_0$ and $g\,{}^{\tau}B_0g^{-1}=B_0$, and then $\mathrm{Int}(g)\sigma$ is an automorphism of $X_*(T_0)$ independent of the choice of g and B_0 and this defines the L-action.

5.2.2. *A* $\mathbf{Q}$*-linearization of the* G*-action: the Hodge filtration.* We will explain now how to construct the ample equivariant line bundle needed to apply the Hilbert-Mumford criterion. We start with a simple but crucial case.

Example 5.2. Let $G = \mathbb{GL}_n$, with diagonal torus T and the standard IIP $\mathscr{P}$ on $X_*(T)_{\mathbf{Q}} = \mathbf{Q}^n$. Let $\mu(t) = (t^{a_1}, \dots, t^{a_1}, \dots, t^{a_r}, \dots, t^{a_r})$, t^{a_i} appearing n_i times and $a_1 > \dots > a_r$ being integers. Then $\mathscr{F}(G, \{\mu\}) = G/P$, P being the upper triangular parabolic with Levi $M = \prod_{i=1}^r \mathbb{GL}_{n_i}$. We define a $G_{\overline{k}}$-equivariant line bundle on $\mathscr{F}_{\overline{k}}$ by

$$\mathscr{L}_{G,\{\mu\},\mathscr{P}} = G \times^P \mathbb{G}_{a,\lambda},$$

where $\lambda \in X^*(M) = X^*(P)$ is defined by $\lambda(g_1, \dots, g_r) = \prod_{i=1}^r (\det g_i)^{-a_i}$ for $(g_1, \dots, g_r) \in M$. The minus sign appears here to get an ample line bundle. We can interpret this geometrically as follows. Let $V = k^n$. Then $\mathscr{F} := \mathscr{F}(G, \{\mu\})$ is the variety (over k) of partial flags of V, of type $(n_1, n_1 + n_2, \dots, n_1 + \dots + n_r = n)$ and so $\mathscr{F}$ is naturally a closed subvariety $\mathscr{F} \hookrightarrow \prod_{i=1}^r X_i$ of a product of Grassmanians $X_i := \mathbb{Gr}_{n_1 + \dots + n_i}(V)$.. Each X_i has a natural very ample $\mathbb{GL}(V)$-equivariant line bundle $\mathscr{L}_i$ obtained from $\mathscr{O}(1)$ via the Plücker embedding $X_i \hookrightarrow \mathbb{P}(\wedge^{n_1 + \dots + n_i}(V))$. Then $\mathscr{L}_{G,\{\mu\},\mathscr{P}}$ is isomorphic (as equivariant line bundle) to the restriction of $\mathscr{L}_1^{\otimes(a_1 - a_2)} \boxtimes \mathscr{L}_2^{\otimes(a_2 - a_3)} \boxtimes \dots \boxtimes \mathscr{L}_r^{\otimes a_r}$. If $x \in \mathscr{F}(K)$ is a point corresponding to a filtration F_x of V_K, then the fiber of $\mathscr{L} = \mathscr{L}_{G,\{\mu\},\mathscr{P}}$ at x is $\otimes_i \det(\mathrm{gr}^{a_i}(F_x))^{-\otimes a_i}$. We deduce immediately from this (see [17, Lemma 2.2.1, Lemma 2.2.2]) that for any point $x \in \breve{\mathscr{F}}(K)$ (K being an extension of $\breve{E}$), and any $\lambda \in X_*(G)^{\mathscr{G}_F}$ with associated filtration F_λ, we have

$$\mu^{\mathscr{L}}(x, \lambda) = -\langle F_x, F_\lambda \rangle.$$

In particular the Hilbert-Mumford criterion is nothing but Corollary 4.1 in this special case.

Proposition 5.3. *Given a connected reductive* k*-group* G*, an element* $\{\mu\} \in X_*(G)_{\mathbf{Q}}/G$ *and an IIP* $\mathscr{P}$ *on* G*, one can naturally construct a* $\mathbf{Q}$*-line bundle*

$$\mathscr{L}_{G,\{\mu\},\mathscr{P}} \in \mathrm{Pic}^{G,\mathrm{ample}}(\mathscr{F}(G, \{\mu\}))_{\mathbf{Q}}$$

defined over the reflex field of $(G, \{\mu\})$ *and such that:*

1. *If* $\iota : G \to G'$ *is a closed immersion defined over* k *and* $\mathscr{P}$ *is induced by an IIP* $\mathscr{P}'$ *on* G'*, then* $\mathscr{L}_{G,\{\mu\},\mathscr{P}}$ *is the restriction of* $\mathscr{L}_{G',\iota\{\mu\},\mathscr{P}'}$ *via the closed immersion* $\mathscr{F}(G, \{\mu\}) \hookrightarrow \mathscr{F}(G', \iota\{\mu\}) \otimes_{E'} E$*, where* E, E' *are the corresponding reflex fields.*
2. *If* $G = \mathbb{GL}_n$ *and* $\mathscr{P}$ *is the standard invariant inner product,* $\mathscr{L}_{G,\{\mu\},\mathscr{P}}$ *is the one in Example 5.2.*

Proof. This follows from the discussion preceding [17, Th. 6.2.8] (for example, [17, Lemma 6.2.5]). The key point is that an IIP $\mathscr{P}$ on G induces one on the Levi quotient M of any parabolic subgroup P of G in a natural way (since any maximal torus of M is an isomorphic image of a maximal torus of G contained in P), which allows one to associate to any $\lambda \in X_*(G)$ an element $\lambda^* \in X^*(P(\lambda))_{\mathbf{Q}}$. The recipe is then induced by sending λ to $\mathscr{L}_\lambda = G \times^P \mathbb{G}_{a,-\lambda^*}$, the G-equivariant $\mathbf{Q}$-line bundle on $G/P(\lambda)$ corresponding to $-\lambda^*$, the minus sign being chosen to ensure that $\mathscr{L}_\lambda$ is ample. □

5.2.3. *A* $\mathbf{Q}$*-linearization of the* J*-action: the Hodge and the slope filtrations.* Let now $k = \mathbf{Q}_p$ and consider a local shtuka datum $(G, [b], \{\mu\})$ over $F = \mathbf{Q}_p$. Fix an IIP $\mathscr{P}$ on G and a *decent* $b \in G(\breve{\mathbf{Q}}_p)$, with associated automorphism group $J = J_b$ and slope morphism ν_b. The element $\lambda_b := -\nu_b \in \mathrm{Hom}_{\breve{\mathbf{Q}}_p}(\mathbb{D}_{\breve{\mathbf{Q}}_p}, G_{\breve{\mathbf{Q}}_p})$ gives rise to an ample $G_{\breve{\mathbf{Q}}_p}$-equivariant $\mathbf{Q}$-line bundle $\mathscr{L}_b := \mathscr{L}_{G,\{\lambda_b\},\mathscr{P}}$ on

$$\mathscr{F}^b := \mathscr{F}(G_{\breve{\mathbf{Q}}_p}, \{\lambda_b\}) = G_{\breve{\mathbf{Q}}_p}/P(\lambda_b),$$

which will be considered as a $J_{\breve{\mathbf{Q}}_p}$-equivariant line bundle via the natural map $J_{\breve{\mathbf{Q}}_p} \to G_{\breve{\mathbf{Q}}_p}$. By the same token we consider $\mathscr{L}_{G,\{\mu\},\mathscr{P}}$ as a $J_{\breve{E}}$-equivariant $\mathbf{Q}$-line bundle on $\mathscr{F}_{\breve{E}}$ and define

$$\mathscr{L}_{G,[b],\{\mu\},\mathscr{P}} := i^*(\mathscr{L}_{G,\{\mu\},\mathscr{P}} \times \mathscr{L}_b) \in \mathrm{Pic}^{J_{\breve{E}},\mathrm{ample}}(\mathscr{F}_{\breve{E}})_{\mathbf{Q}},$$

the closed embedding $i : \mathscr{F}_{\breve{E}} \hookrightarrow \mathscr{F}_{\breve{E}} \times \mathscr{F}^b_{\breve{E}}$ being given by the identity on the first factor and by the $\breve{E}$-rational point λ_b of $\mathscr{F}^b_{\breve{E}}$ on the second factor. The construction enjoys similar properties to the one from Proposition 5.3, concerning functoriality with respect to closed embeddings $\iota : G \to G'$ (by taking $b' = \iota(b)$, of course).

5.2.4. *A Hilbert-Mumford criterion for weak admissiblity.* We keep the notations introduced in the previous paragraph and set $\mathscr{L} = \mathscr{L}_{G,b,\{\mu\},\mathscr{P}} \in \mathrm{Pic}^{J_{\breve{E}},\mathrm{ample}}(\breve{\mathscr{F}})_{\mathbf{Q}}$, where $\breve{\mathscr{F}} := \mathscr{F}(G,\{\mu\})_{\breve{E}}$.

Theorem 5.4 (Totaro, [66, Th. 3], [17, Th. 9.7.3])**.** *Let $K/\breve{E}$ be a field extension and let $x \in \breve{\mathscr{F}}(K)$. Then $x \in \breve{\mathscr{F}}(G, b, \{\mu\})^{\mathrm{wa}}(K)$ if and only if $\mu^{\mathscr{L}}(x, \lambda) \geq 0$ for all $\lambda \in X_*(J)^{\mathscr{G}_F}$.*

Example 5.5. Suppose that $G = T$ is a torus over $\mathbf{Q}_p$. We then have $J = T$ and

$$\mu^{\mathscr{L}}(x, \lambda) = -(\mathscr{P}(\lambda, \lambda_x) + \mathscr{P}(\lambda, \nu_b)),$$

for any IIP $\mathscr{P}$ on T, any $\lambda \in X_*(T)$, and $\lambda_x \in X_*(T)$ attached to x. Hence the Hilbert-Mumford inequality takes the form: $\mathscr{P}(\lambda, \lambda_x) + \mathscr{P}(\lambda, \nu_b) \leq 0$ for all $\mathbf{Q}_p$-rational 1-PS λ of T. Since we may replace λ with $-\lambda$ this condition is satisfied if and only if $\lambda_x + \nu_b$ is orthogonal to all $\mathbf{Q}_p$-rational 1-PS of T. Since the inner product is $\mathscr{G}_{\mathbf{Q}_p}$-invariant, the $\mathbf{Q}_p$-rational characters of T correspond to the $\mathbf{Q}_p$-rational 1-PS and the Hilbert-Mumford criterion becomes the familiar criterion: (μ, b) is weakly admissible if and only if $\mu + \nu_b$ is orthogonal to all $\mathbf{Q}_p$-rational characters of T.

Example 5.6. Consider the set-up from Example 5.2. The data of b is equivalent to that of an isocrystal $N = (V, \varphi)$ of dimension n over $\breve{\mathbf{Q}}_p$, with slope decomposition $V = \oplus_{\alpha \in \mathbf{Q}} V_\alpha$. Let $\alpha_1 > \cdots > \alpha_t$ be the different slopes and let $m_i = \dim V_{\alpha_i}$, so that the Newton vector of N is $\nu(N) = (\alpha_1^{(m_1)}, \ldots, \alpha_t^{(m_t)}) \in (\mathbf{Q}^n)_+$. Consider the filtration F_0 defined by $F_0^\beta = \oplus_{\alpha \leq -\beta} V_\alpha$ (its type is thus $\nu_0 = (-\alpha_t^{(m_t)}, \ldots, -\alpha_1^{(m_1)}) \in (\mathbf{Q}^n)_+$). An immediate computation [17, Lemma 8.4.2] shows that if $K/\breve{\mathbf{Q}}_p$ is an extension and $x \in \breve{\mathscr{F}}(K)$ with corresponding filtration F_x of V_K, then for all $\lambda \in X_*(J)^{\mathscr{G}_F}$

$$\mu^{\mathscr{L}}(x, \lambda) = -(\langle F_x, F_\lambda \rangle + \langle F_0, F_\lambda \rangle).$$

The Hilbert-Mumford inequality $\mu^{\mathscr{L}}(x, \lambda) \geq 0$ is thus equivalent to $\langle F_x, F_\lambda \rangle + \langle F_0, F_\lambda \rangle \leq 0$.

The computations in Example 5.6 and the basic properties of the slope function recalled in Section 5.1 yield the first part of the following lemma. The second part is an immediate calculation.

Lemma 5.7 (Orlik, [51, Lemma 2.2], [50, Lemma 2.2])**.** *Let V be a faithful F-rational representation of G (defined over F) and consider the IIP $\mathscr{P}$ on G induced by the standard IIP on $\mathbb{GL}(V)$.*

(1) *Let $K/\breve{E}$ be an extension, $x \in \breve{\mathscr{F}}(K)$ and $\lambda \in X_*(J)^{\mathscr{G}_F}$. Let F_x, F_λ, and F_0 denote the filtrations on $V_{\breve{E}}$ induced by x, λ and $\lambda_b = -\nu_b$, respectively. Then*

$$\mu^{\mathscr{L}}(x, \lambda) = -(\langle F_x, F_\lambda \rangle + \langle F_0, F_\lambda \rangle).$$

(2) *If $T \subset G$ is a maximal torus and $\lambda, \lambda' \in X_*(T)_{\mathbf{Q}}$, then $\langle F_\lambda, F_{\lambda'} \rangle = \mathscr{P}(\lambda, \lambda')$.*

5.3. **A simplified Hilbert-Mumford criterion.** We will show here that in the Hilbert-Mumford criterion from Theorem 5.4 it suffices to test the 1-PS associated to the relative simple roots and their conjugates. This will follow from the fact that the slope function, a priori just convex on every chamber of the spherical building, is, in fact, affine.

5.3.1. *The criterion. We will assume from now on that $(G, [b], \{\mu\})$ is a local Shtuka datum over $F = \mathbf{Q}_p$ with G quasi-split over $\mathbf{Q}_p$ and $b \in [b]$ basic and s-decent.* Let $J = J_b$, $\nu = \nu_b$, let $E = E(G, \{\mu\})$ be the reflex field and let $E_s = E\mathbf{Q}_{p^s}$. The associated period domain has a canonical model over E_s,invariant under the natural action of $J(\mathbf{Q}_p)$ on $\mathscr{F}$.

We fix an invariant inner product $\mathscr{P}$ on G and we note that it gives rise to an invariant inner product on J. Fix a maximal $\mathbf{Q}_p$-split torus $S \subset J_{\mathrm{der}}$, of $\mathbf{Q}_p$-rank d, and a minimal parabolic subgroup P_0 of J defined over $\mathbf{Q}_p$. Let $\Delta = \{\alpha_1, \dots, \alpha_d\} \subset X^*(S)$ be the corresponding set of relative simple roots. Let $\omega_{\alpha_1}, \dots, \omega_{\alpha_d} \in X_*(S)_{\mathbf{Q}}$ be the associated dual basis for the natural pairing between $X^*(S)_{\mathbf{Q}}$ and $X_*(S)_{\mathbf{Q}}$. Fix a maximal torus T of G containing S and such that $\mu, \nu \in \operatorname{Hom}_{\breve{\mathbf{Q}}_p}(\mathbb{D}, T) \simeq X_*(T)_{\mathbf{Q}}$.

Proposition 5.8 (Orlik, [50, Cor. 2.4])**.** *Let $x \in \breve{\mathscr{F}}(K)$, for a field extension K of $\breve{E}$. Then x is not weakly admissible if and only if there exists an element $g \in J(\mathbf{Q}_p)$ and a simple root $\alpha \in \Delta$ such that $\mu^{\mathscr{L}}(x, \operatorname{Int}(g)\omega_\alpha) < 0$.*

5.3.2. *The combinatorial and spherical buildings.* If G is a connected reductive group over a field k, one can associate to G two buildings, as follows.

The *combinatorial building* $\Delta(G)$ of G. An abstract simplicial complex with $G(k)$-action whose simplices are the proper k-parabolic subgroups of G ordered by the opposite of inclusion (thus vertices are the proper maximal k-parabolics and $(P_0, \dots, P_d)$ is a simplex if and only if $P_0 \cap \cdots \cap P_d$ is a parabolic subgroup). If n denotes the k-rank of the derived group of G then $\Delta(G)$ has the homotopy type of a bouquet of $(n-1)$-spheres.

The *spherical building* $\mathscr{B}(G)$ of G. Unlike $\Delta(G)$ it takes into account the center of G. If m is the k-rank of G the set $\mathscr{B}(G)$ is, in a $G(k)$-equivariant way, the $(m-n)$-fold suspension of $\Delta(G)$. Thus suitably topologized, it has the homotopy type of a bouquet of $(m-1)$-spheres. The building $\mathscr{B}(G)$ is functorial for injective maps of reductive groups $f : G \to H$ over k: we have a natural embedding of topological spaces $\mathscr{B}(f) : \mathscr{B}(G) \to \mathscr{B}(H)$.

If $G = S$ is a split k-torus, then $\mathscr{B}(S)$ is simply the sphere corresponding to half-lines in the vector space $X_*(S)_{k,\mathbf{R}}$. In general, $\mathscr{B}(G)$ is obtained by gluing the different spheres $\mathscr{B}(S)$ (over all maximal k-split tori $S \subset G$), where we identify b and $\operatorname{Int}(g)b$ for $b \in \mathscr{B}(S)$ and $g \in P(b)(k)$. Here $P(b)$ is a k-parabolic of G naturally attached to b in a way compatible with the definition of $P(\lambda)$, for $\lambda \in X_*(S)_{\mathbf{Q}}$. There is a natural action of $G(k)$ on $\mathscr{B}(G)$ and a natural map $b \to P(b)$ from $\mathscr{B}(G)$ to the set of k-parabolic subgroups of G (we note $P(b)(k)$ is the stabilizer of b in $G(k)$). Moreover, if S is a maximal k-split torus of G, the map $\mathscr{B}(S) \to \mathscr{B}(G)$ is injective and consists precisely of points b of $\mathscr{B}(G)$ for which $S \subset P(b)$. We call the image the apartment attached to S. Any two points of $\mathscr{B}(G)$ belong to a common apartment.

Assume that G is semisimple. Then, by a theorem of Curtis, Lehrer and Tits [15, Prop. 6.1], there is a natural $G(k)$-equivariant bijection $\tau : |\Delta(G)| \to \mathscr{B}(G)$ between the geometric realization of $\Delta(G)$ and $\mathscr{B}(G)$, such that for $b \in |\Delta(G)|$, the k-parabolic $P(\tau(b))$ is the one corresponding to the simplex of $\Delta(G)$ containing b in its interior. Hence the combinatorial building yields a triangulation of the spherical building.

For a k-rational parabolic subgroup $P \subset G$, we set

$$\mathscr{C}(P) := \{b \in \mathscr{B}(G) : P(b) \supset P\}.$$

We have $\mathscr{C}(P) \subset \mathscr{B}(S)$, where S is a maximal k-split torus contained in P. We can think of $\mathscr{C}(P)$ as parametrizing dominant 1-PS λ of P up to conjugation and ramification $\lambda \mapsto \lambda^n$. The map τ induces a homeomorphism between the closed simplex of $|\Delta(G)|$ corresponding to P and $\mathscr{C}(P)$. If P is a minimal k-parabolic than $\mathscr{C}(P)$ is called a *chamber*; if P is a proper maximal k-parabolic subgroup then $\mathscr{C}(P)$ is called a vertex.

5.3.3. *The slope function revisited.* We return to the notation from Section 5.3.1. Let S be a maximal $\mathbf{Q}_p$-split torus in J_{der}. Fix $x \in \mathscr{F}(K)$, for a field extension K of $\breve{E}$. We can extend the slope function $\mu^{\mathscr{L}}(x, \lambda)$ on $X_*(S)_{\mathbf{Q}}$ to a function on $X_*(S)_{\mathbf{R}}$ by using its description as an infimum of values of certain rational linear functionals [56, Sec. 1]. This description also implies that the slope function $\mu^{\mathscr{L}}(x, \lambda)$ is convex on the rational points of $\mathscr{B}(S)$ [48, Cor. 2.15].

It turns out, as shown by Orlik, that the slope function is, in fact (in a suitable sense), affine. To state precisely what this means, for a chamber $\mathscr{C}$ in $\mathscr{B}(J_{\mathrm{der}})$, we start with "straightening it", i.e., we deform it homeomorphically to the simplex

$$\widetilde{w}\mathscr{C} := \{\sum_{\alpha\in\Delta} r_\alpha\lambda_\alpha | 0 \leq r_\alpha \leq 1, \sum_{\alpha\in\Delta} r_\alpha = 1\} \subset X_*(\widetilde{w}S)_{\mathbf{R}},$$

where the rational 1-PS $\lambda_\alpha \in X_*(\widetilde{w}S)_{\mathbf{Q}}$, $\alpha \in \Delta$, for some maximal $\mathbf{Q}_p$-split torus $\widetilde{w}S \subset J_{\rm der}$, represent the vertices of $\mathscr{C}$. What we have gained doing this is that the slope function $\mu^{\mathscr{L}}(x,\lambda)$ on $X_*(\widetilde{w}S)_{\mathbf{R}}$ while needing to be normalized to $\nu^{\mathscr{L}}(x,\lambda) := \mu^{\mathscr{L}}(x,\lambda)/|\lambda|$ to descend to a function on $\mathscr{B}(\widetilde{w}S)$ (after which it glues to a function on $\mathscr{B}(J_{\rm der})$) is defined on $\widetilde{w}\mathscr{C}$. We say that $\mu^{\mathscr{L}}(x,-)$ is *affine* on $\mathscr{C}$ if it is affine on $\widetilde{w}\mathscr{C}$, i.e., we have:

$$\mu^{\mathscr{L}}(x, \sum_{\alpha\in\Delta} r_\alpha\lambda_\alpha) = \sum_{\alpha\in\Delta} r_\alpha\mu^{\mathscr{L}}(x,\lambda_\alpha), \quad \text{for all } \sum_{\alpha\in\Delta} r_\alpha\lambda_\alpha \in \widetilde{w}\mathscr{C}.$$

The following lemma is now a simple consequence of Lemma 5.7 that describes the slope function via the invariant inner product (which is linear in each variable):

Lemma 5.9 (Orlik, [50, Prop. 2.3])**.** *Let $x \in \mathscr{F}(K)$, for a field extension K of $\breve{E}$. The slope function $\mu^{\mathscr{L}}(x,-)$ is affine on each chamber of $\mathscr{B}(J_{\rm der})$.*

5.3.4. *Proof of Proposition 5.8.*

Proof. Among the chambers of $\mathscr{B}(J_{\rm der})$ we will distinguish the *base chamber* $\mathscr{C}_0 := \mathscr{C}(P_0)$. We note here that $\Delta(J) = \Delta(J_{\rm der})$, hence $\mathscr{B}(J_{\rm der})$ is homeomorphic to $|\Delta(J)|$. The vertices of this chamber are the rational 1-PS ω_α, $\alpha \in \Delta$; the corresponding parabolic subgroups [24] $P^J(\omega_\alpha)$, $\alpha \in \Delta$, are the maximal $\mathbf{Q}_p$-rational parabolic subgroups that contain P_0. If $\mathscr{C} = \mathscr{C}(P)$ is a chamber in $\mathscr{B}(J_{\rm der})$ there exists a $g \in J(\mathbf{Q}_p)$, unique up to multiplication by an element of $P_0(\mathbf{Q}_p)$ from the right, such that the conjugated 1-PS $\mathrm{Int}(g)\omega_\alpha$, $\alpha \in \Delta$, are the vertices of $\mathscr{C}$.

Proposition 5.8 follows now easily from Lemma 5.9. ☐

5.3.5. *Contractibility of a subcomplex of the spherical building.* Let

$$Y := \mathscr{F} \otimes_E E_s \setminus \mathscr{F}^{\rm wa}$$

be the complement of $\mathscr{F}^{\rm wa}$. Let $x \in Y(K)$, for a field extension K of $\breve{E}$. Consider the subcomplex T_x of the spherical buidling $\mathscr{B}(J_{\rm der})$ corresponding to the following set of vertices:

$$\{gP(\omega_\alpha)g^{-1} | g \in J(\mathbf{Q}_p), \alpha \in \Delta \text{ such that } \mu^{\mathscr{L}}(x, \mathrm{Int}(g)\omega_\alpha) < 0\}.$$

We will need to know that this subcomplex is contractible. This follows from two facts:

(1) Let
$$C_x := \{\lambda \in \mathscr{B}(J_{\rm der}) | \nu^{\mathscr{L}}(x,\lambda) < 0\}.$$
This set is convex and the intersection of C_x with each chamber in $\mathscr{B}(J_{\rm der})$ is convex [48, Cor. 2.16].

(2) Because the slope function $\mu^{\mathscr{L}}(x,\lambda)$ is affine on every chamber of $\mathscr{B}(J_{\rm der})$ we have an inclusion $T_x \hookrightarrow C_x$. This is a deformation retract [50, Lemma 3.4] (in fact, we deform to a projection).

---

[24] We have $P^J(\omega_\alpha)(\overline{\mathbf{Q}}_p) = P(\omega_\alpha)(\overline{\mathbf{Q}}_p) \cap J(\overline{\mathbf{Q}}_p)$.

5.4. **Stratification of the complement of $\mathscr{F}^{\mathrm{wa}}$.** The complement Y of $\mathscr{F}^{\mathrm{wa}}$ in $\mathscr{F}_{E_s}$ can be stratified using Schubert varieties, an essential result for the computation of its cohomology. In order to describe this stratification we will fully use the results presented above (we keep the notation from Section 5.3.1).

Each $\lambda \in X_*(J)_{\mathbf{Q}}$ of J determines a closed subvariety

$$Y_\lambda := \{x \in \mathscr{F} \,|\, \mu^{\mathscr{L}}(x, \lambda) < 0\}$$

of $\mathscr{F}$, defined over E_s, consisting of points where λ damages the semistability condition. In particular, each subset $I \subsetneq \Delta$ gives rise to a closed subvariety of $\mathscr{F}$, defined over E_s

$$Y_I := \bigcap_{\alpha \notin I} Y_{\omega_\alpha}$$

and the properties of the slope function imply that the natural action of $J(\mathbf{Q}_p)$ on $\mathscr{F}_{E_s}$ restricts to an action of $P_I(\mathbf{Q}_p)$ on Y_I [51, Lemma 3.1], where $P_I = \cap_{\alpha \notin I} P^J(\omega_\alpha)$. For example $P_\Delta = J$ and $P_\varnothing = P_0$, a fixed minimal $\mathbf{Q}_p$-parabolic subgroup of J containing S.

Let

$$X_I = J(\mathbf{Q}_p)/P_I(\mathbf{Q}_p),$$

a compact p-adic analytic manifold, and let[25]

$$Z_I^{X_I} = \bigcup_{t \in X_I} tY_I^{\mathrm{ad}},$$

a closed pseudo-adic subspace[26] of Y (since X_I is compact, see [51, Lemma 3.2]). Now, by Proposition 5.8 and the properties of the slope function $\mu^{\mathscr{L}}(-,-)$ (see Section 5.1), we get the decomposition

$$Y = \bigcup_{|\Delta \setminus I| = 1} Z_I^{X_I}.$$

5.5. **The cohomology of Schubert varieties.** Orlik used the results from GIT described in the previous sections to show (see [51, Prop. 4.1], [50, Prop. 4.1]) that the varieties Y_I are *Schubert varieties*, hence their cohomology is not difficult to compute.

More precisely, fix a Borel subgroup B of G contained in all the parabolics $P(\omega_\alpha)$ and such that μ belongs to the positive Weyl chamber with respect to B. Let $W = N_G(T)/T$ be the Weyl group, let $W_\mu \subset W$ be the stabilizer of μ and let W^μ be the set of Kostant representatives for W/W_μ, i.e., representatives of shortest length in their cosets. The action of $\mathscr{G}_{E_s}$ on W preserves W^μ (since μ is defined over E_s). For $w \in W$, let $[w] \in W^\mu/\mathscr{G}_{E_s}$ be its orbit.

Define

$$\Omega_I = \{[w] \in W^\mu/\mathscr{G}_{E_s} \,|\, \mathscr{P}(w\mu - \nu, \omega_\alpha) > 0,\ \forall \alpha \notin I\},$$

where $\mathscr{P}$ is the fixed invariant inner product on G. Then $\Omega_{I\cap J} = \Omega_J \cap \Omega_J$ and a similar property holds for the varieties Y_I. The Bruhat decomposition of the flag variety $\mathscr{F}_{E_s}$ combined with the fact that, by Lemma 5.7, the semistability criterion can be expressed using the invariant inner product, yield the decomposition of Y_I in terms of the Bruhat cells of G with respect to $P(\mu)$:

$$Y_I = \bigcup_{[w] \in \Omega_I} BwP(\mu)/P(\mu). \tag{5.10}$$

To each $\mathscr{G}_{E_s}$-orbit $[w] \in W^\mu/\mathscr{G}_{E_s}$ we associate the following objects:

- An integer $l_{[w]}$, the length of any element of $[w]$.
- The induced $\mathbf{Z}/p^n[\mathscr{G}_{E_s}]$-module $\rho_{[w]}(\mathbf{Z}/p^n)$ of $\mathbf{Z}/p^n$-valued functions on the finite set $[w]$, with the natural $\mathscr{G}_{E_s}$-action twisted (à la Tate) by $-l_{[w]}$. Similarly, we define a $\mathbf{Z}_p[\mathscr{G}_{E_s}]$-module $\rho_{[w]}(\mathbf{Z}_p)$.

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[25] The definition makes sense since $P_I(\mathbf{Q}_p)$ preserves Y_I.

[26] We refer the reader to the appendix for the formalism of pseudo-adic spaces, due to Huber [36].

The Bruhat decomposition (5.10) yields the following computation of the étale cohomology of the varieties Y_I.

Corollary 5.11. *We have*

$$H^*_{\text{ét}}(Y_{I,C},\mathbf{Z}/p^n)\simeq\bigoplus_{[w]\in\Omega_I}\rho_{[w]}(\mathbf{Z}/p^n)[-2l_{[w]}]. \tag{5.12}$$

Proof. The computation in the ℓ-adic setting in [50, Prop. 4.2] [49, Prop. 7.1] carries over to the p-adic setting, the key element being the standard form of the compactly supported cohomology of the affine space.

The computation goes as follows. For $0\leq i\leq m_I:=\max\{l_{[w]}|[w]\in\Omega_I\}$, set

$$Y_I^i:=\bigcup_{[w]\in\Omega_I,l(w)\leq i}BwP(\mu)/P(\mu).$$

We have a filtration by closed subvarieties

$$Y_I=Y_I^{m_I}\supset Y_I^{m_I-1}\supset\cdots\supset Y_I^0\supset Y_I^{-1}:=\varnothing$$

such that

$$Y_I^i\setminus Y_I^{i-1}=\coprod_{[w]\in\Omega_I,l(w)=i}BwP(\mu)/P(\mu). \tag{5.13}$$

The Bruhat cell $BwP(\mu)/P(\mu)$ above is isomorphic to the affine space $\mathbb{A}^i_{E_s}$. Recall that

$$H^j_{\text{ét},c}(\mathbb{A}^i_C,\mathbf{Z}/p^n)\simeq\begin{cases}\mathbf{Z}/p^n(-i) & \text{for } j=2i,\\ 0 & \text{otherwise.}\end{cases}$$

The corollary follows now easily from the long exact sequences (the induced Galois representations arise from the Bruhat decomposition (5.13))

$$\cdots\to H^{i-1}_{\text{ét},c}(Y^{j-1}_{I,C},\mathbf{Z}/p^n)\to H^i_{\text{ét},c}(Y^j_{I,C}\setminus Y^{j-1}_{I,C},\mathbf{Z}/p^n)\to H^i_{\text{ét},c}(Y^j_{I,C},\mathbf{Z}/p^n)\to H^i_{\text{ét},c}(Y^{j-1}_{I,C},\mathbf{Z}/p^n)\to\cdots$$

□

Remark 5.14. The following observation will be useful later (see [50, Sec. 4]). For $[w]\in W^\mu/\mathscr{G}_{E_s}$ and $I\subset\Delta$, let $H(Y_I,[w])$ denote the part of the direct sum (5.12), which comes from $[w]$, i.e.,

$$H(Y_I,[w])=\begin{cases}\rho_{[w]}(\mathbf{Z}/p^n)[-2l_{[w]}] & \text{if } [w]\in\Omega_I,\\ 0 & \text{if } [w]\notin\Omega_I.\end{cases}$$

We have

$$H^*_{\text{ét}}(Y_{I,C},\mathbf{Z}/p^n)\simeq\bigoplus_{[w]\in W^\mu/\mathscr{G}_{E_s}}H(Y_I,[w]).$$

Let $I\subset J\subset\Delta$. Consider the projections

$$p_{I,J}:H^*_{\text{ét}}(Y_{J,C},\mathbf{Z}/p^n)\to H^*_{\text{ét}}(Y_{I,C},\mathbf{Z}/p^n)$$

induced by the closed embeddings $Y_I\hookrightarrow Y_J$. The proof of Corollary 5.11 shows that they decompose into the direct sums:

$$p_{I,J}\simeq\bigoplus_{([w],[w'])\in(W^\mu/\mathscr{G}_{E_s})^2}p_{[w],[w']}:\bigoplus_{[w]\in W^\mu/\mathscr{G}_{E_s}}H(Y_J,[w])\to\bigoplus_{[w']\in W^\mu/\mathscr{G}_{E_s}}H(Y_I,[w'])$$

with $p_{[w],[w']}$ equal to the identity for $[w]=[w']$ and to zero otherwise.

6. THE MAIN RESULT

We are now ready to formulate and to prove the main result of this paper.

6.1. **The setup.** Consider a local Shtuka datum $(G, [b], \{\mu\})$ over $F = \mathbf{Q}_p$, where G is a *quasi-split* reductive group over $\mathbf{Q}_p$, $[b] \in B(G)$ is the σ-conjugacy class of a *basic* and s-decent[27] element $b \in G(\breve{\mathbf{Q}}_p)$, and $\{\mu\}$ is a conjugacy class of cocharacters of G, with field of definition E and associated flag variety $\mathscr{F} = \mathscr{F}(G, \{\mu\})$ defined over E.

Letting $E_s = E\mathbf{Q}_{p^s} \subset \overline{\mathbf{Q}}_p$, the period domain $\mathscr{F}^{\mathrm{wa}} = \mathscr{F}^{\mathrm{wa}}(G, [b], \{\mu\})$ is a nonempty[28] partially proper open subset of the adic space $\mathscr{F}_{\breve{E}}$, stable under the natural action of $J(\mathbf{Q}_p) \subset G(\breve{\mathbf{Q}}_p)$ on $\mathscr{F}_{\breve{E}}$ and having a canonical model over E_s. Thus the compactly supported étale cohomologies $H^*_{\text{ét},c}(\mathscr{F}^{\mathrm{wa}}_C, \mathbf{Z}/p^n)$ and $H^*_{\text{ét},c}(\mathscr{F}^{\mathrm{wa}}_C, \mathbf{Z}_p)$ are naturally $J(\mathbf{Q}_p) \times \mathscr{G}_{E_s}$-modules. Our goal is to describe these representations.

6.1.1. *Generalized Steinberg representations.* Fix a maximal $\mathbf{Q}_p$-split torus S in $J = J_b$ and a minimal parabolic subgroup B of $J = J_b$ defined over $\mathbf{Q}_p$ and having Levi component the centralizer of S. This choice induces a parametrization $I \mapsto P_I$ of the standard parabolic subgroups of J by subsets of Δ, the set of simple roots associated to B. In particular $P_\varnothing = B$ and $P_\Delta = J$.

We slightly change the notation introduced in the first section and write simply

$$X_I = J(\mathbf{Q}_p)/P_I(\mathbf{Q}_p), \quad i_I(A) = \mathrm{LC}(X_I, A), \quad A = \mathbf{Z}/p^n, \mathbf{Z}_p, \mathbf{Q}_p,$$

where $\mathrm{LC}(X_I, A)$ is the space of locally constant (automatically with compact support since X_I is compact) functions on X_I with values in A. We also let

$$i_I^{\mathrm{cont}}(\mathbf{Z}_p) := \mathscr{C}(X_I, \mathbf{Z}_p) = \varprojlim_n i_I(\mathbf{Z}/p^n),$$

where $\mathscr{C}(X_I, \mathbf{Z}_p)$ is the space of continuous functions on X_I with $\mathbf{Z}_p$. The associated generalized smooth and continuous Steinberg representations of $J(\mathbf{Q}_p)$ will be denoted

$$v_I(A) := i_I(A)/\sum_{I \subsetneq I'} i_{I'}(A), \quad A = \mathbf{Z}/p^n, \mathbf{Z}_p, \mathbf{Q}_p,$$
$$v_I^{\mathrm{cont}}(\mathbf{Z}_p) := i_I^{\mathrm{cont}}(\mathbf{Z}_p)/\sum_{I \subsetneq I'} i_{I'}^{\mathrm{cont}}(\mathbf{Z}_p) = \varprojlim_n v_I(\mathbf{Z}/p^n),$$

where the last equality follows from [30, Lemme 3.3.3] (in loc. cit. J is split, but the results extend verbatim to any J). Note that $i_\Delta = v_\Delta$ is simply the trivial representation of $J(\mathbf{Q}_p)$.

For two subsets $I \subset I' \subset \Delta$ with $|I' \setminus I| = 1$, we let $p_{I,I'} : i_{I'}(\mathbf{Z}/p^n) \to i_I(\mathbf{Z}/p^n)$ be the natural map induced by the surjection $X_I \to X_{I'}$. For arbitrary subsets $I, I' \subset \Delta$ with $|I'| - |I| = 1$, we fix a numbering $I' = \{\beta_1, \dots, \beta_r\}$ and we let

$$d_{I,I'} = \begin{cases} (-1)^i p_{I,I'} & \text{if } I' = I \cup \{\beta_i\}, \\ 0 & \text{if } I \not\subset I'. \end{cases}$$

The following result is standard for complex coefficients.

Proposition 6.1. *Let $I \subset \Delta$ and consider the complex whose first term is in degree -1*

$$C_I(\mathbf{Z}/p^n) : i_\Delta(\mathbf{Z}/p^n) \to \bigoplus_{\substack{I \subset K \subset \Delta \\ |\Delta \setminus K| = 1}} i_K(\mathbf{Z}/p^n) \to \bigoplus_{\substack{I \subset K \subset \Delta \\ |\Delta \setminus K| = 2}} i_K(\mathbf{Z}/p^n) \to \cdots$$
$$\cdots \to \bigoplus_{\substack{I \subset K \subset \Delta \\ |K \setminus I| = 1}} i_K(\mathbf{Z}/p^n) \to i_I(\mathbf{Z}/p^n)$$

with differentials induced by the $d_{K,K'}$. Then $C_I(\mathbf{Z}/p^n)$ is a resolution of $v_I(\mathbf{Z}/p^n)$ by $J(\mathbf{Q}_p)$-modules.

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[27] Recall that the hypothesis that b is decent is harmless, since any σ-conjugacy class in $G(\breve{\mathbf{Q}}_p)$ contains a decent element.

[28] By the definition of a local Shtuka datum!

Proof. The proof for complex coefficients goes over in our setup thanks to a multiplicity one result of Grosse-Klönne [28], Herzig [35], and Ly [46], as we explain below.

Recall first the following simple fact from [62, Sec. 2, Prop. 6]: if $(A_1, \ldots, A_m)$ is a family of subgroups of an abelian group A such that

$$\Big(\sum_{i\in T} A_i\Big) \cap \Big(\bigcap_{j\in S} A_j\Big) = \sum_{i\in T}\Big(A_i \cap \Big(\bigcap_{j\in S} A_j\Big)\Big) \tag{6.2}$$

for all subsets $T, S \subset \{1, \ldots, m\}$, then the natural complex

$$A \leftarrow \bigoplus_{1\leq i\leq m} A_i \leftarrow \bigoplus_{1\leq i<j\leq m} A_i \cap A_j \leftarrow \bigoplus_{1\leq i<j<k\leq m} A_i \cap A_j \cap A_k \leftarrow \cdots$$

is a resolution of the subgroup $\sum_i A_i$ of A.

Now assume that the abelian group A has finite length and say that a subgroup $A' \subset A$ is *isotypically closed* if each irreducible constituent of A' has the same multiplicity in A' and A. Such a subgroup is uniquely determined by its irreducible constituents. If two subgroups $A', A'' \subset A$ are isotypically closed, then $A' \cap A''$ and $A' + A''$ are isotypically closed too, and their irreducible constituents are, respectively, the intersection and the union of the irreducible constituents of A' and A''. We deduce that if A_i is isotypically closed for all i, then the left and right sides of (6.2) are isotypically closed and have the same irreducible constituents (as union distributes over intersection), hence they are equal.

We apply this fact with $A = i_B(\mathbf{Z}/p^n)$ and the $J(\mathbf{Q}_p)$-submodules $i_K(\mathbf{Z}/p^n)$ for $I \subset K \subset \Delta$, $|\Delta \setminus K| = 1$. The latter are isotypically closed in $i_B(\mathbf{Z}/p^n)$ by Proposition 6.3 below. ☐

Proposition 6.3 (Grosse-Klönne, Herzig, Ly)**.** *The irreducible constituents of $i_K(\mathbf{Z}/p^n)$ are the representations $(v_I(\mathbf{Z}/p^n))_{K\subset I\subset\Delta}$, each occurring with multiplicity n.*

Proof. By dévissage, the result reduces to the case $n = 1$, which is due to Grosse-Klönne [28] and Herzig [35] when $\boldsymbol{G}$ is split, and to Ly [46] in general. ☐

6.1.2. *The main result.* The following theorem is the principal result of this paper.

Theorem 6.4. *Let $(G, [b], \{\mu\})$ be a local Shtuka datum with $G/\mathbf{Q}_p$ reductive and quasi-split, $b \in G(\breve{\mathbf{Q}}_p)$ basic and s-decent. Suppose that $p \geq 5$.*

1. *There is an isomorphism of $\mathscr{G}_{E_s} \times J(\mathbf{Q}_p)$-modules*

$$H^*_{\text{ét},c}(\mathscr{F}^{\text{wa}}_C, \mathbf{Z}/p^n) \simeq \bigoplus_{[w]\in W^\mu/\mathscr{G}_{E_s}} v_{I_{[w]}}(\mathbf{Z}/p^n) \otimes \rho_{[w]}(\mathbf{Z}/p^n)[-n_{[w]}], \tag{6.5}$$

   where $n_{[w]} = 2l_{[w]} + |\Delta \setminus I_{[w]}|$.

2. *There is an isomorphism of $\mathscr{G}_{E_s} \times J(\mathbf{Q}_p)$-modules*

$$H^*_{\text{ét},c}(\mathscr{F}^{\text{wa}}_C, \mathbf{Z}_p) \simeq \bigoplus_{[w]\in W^\mu/\mathscr{G}_{E_s}} v^{\text{cont}}_{I_{[w]}}(\mathbf{Z}_p) \otimes \rho_{[w]}(\mathbf{Z}_p)[-n_{[w]}]. \tag{6.6}$$

Remark 6.7.

1. Orlik shows in [51, Th. 1.1] that the isomorphism (6.5) holds for compactly supported étale cohomology with coefficients $\mathbf{Z}/\ell^n$, $\ell \neq p$ (there are additional assumptions on ℓ needed, see [51, Sec. 1]). This implies (by the same argument as we use below) the isomorphism (6.6) for compactly supported étale cohomology with coefficients $\mathbf{Z}_\ell$.
2. Just as in [17, Cor. 10.3.7] one obtains strong vanishing results for $H^*_{\text{ét},c}(\mathscr{F}^{\text{wa}}_C, \mathbf{Z}/p^n)$ from the above theorem, as well as a simple description of the top-degree cohomology.

6.1.3. *The case of the Drinfeld space.* Let $p\geq 5$. Let us discuss an example. Let $\mathbb{H}^d_{\mathbf{Q}_p}$ be the Drinfeld symmetric space of dimension d over $\mathbf{Q}_p$. Recall that $\mathbb{H}^d_{\mathbf{Q}_p}=\mathbb{P}^d_{\mathbf{Q}_p}\setminus\cup_{H\in\mathscr{H}}H$, where $\mathscr{H}$ is the set of $\mathbf{Q}_p$-rational hyperplanes. Set $G:=\mathbb{GL}_{d+1,\mathbf{Q}_p}$.

Corollary 6.8. (1) *There is an isomorphism of $\mathscr{G}_{\mathbf{Q}_p}\times G(\mathbf{Q}_p)$-modules*

$$H^*_{\text{ét},c}(\mathbb{H}^d_C,\mathbf{Z}/p^n)\simeq\bigoplus_{i=0}^{d}\mathrm{Sp}_{d-i}(\mathbf{Z}/p^n)(-i)[-d-i].$$

(2) *There is an isomorphism of $\mathscr{G}_{\mathbf{Q}_p}\times G(\mathbf{Q}_p)$-modules*

$$H^*_{\text{ét,c}}(\mathbb{H}^d_C,\mathbf{Z}_p)\simeq\bigoplus_{i=0}^{d}\mathrm{Sp}^{\text{cont}}_{d-i}(\mathbf{Z}_p)(-i)[-d-i].$$

Here the generalized Steinberg representations $\mathrm{Sp}_{d-i}(\mathbf{Z}/p^n)$ and $\mathrm{Sp}^{\text{cont}}_{d-i}(\mathbf{Z}_p)$ are as defined in the proof.

Proof. By Example 4.2, $\mathbb{H}^d_{\mathbf{Q}_p}$ is the period domain corresponding to $b=1,\{\mu\}=(d,(-1)^d\}$. We have $E=E_s=\mathbf{Q}_p$ and $J=G$.

Let B be the upper triangular Borel subgroup of G and let $\Delta=\{\alpha_1,\alpha_2,\ldots,\alpha_d\}$ be the set of relative simple roots: $\alpha_i(\mathrm{diag}(t_1,\ldots,t_{d+1}))=t_it_{i+1}^{-1}$. We identify the Weyl group W of G with the group of permutations of $\{1,2,\ldots,d+1\}$ and with the subgroup of permutation matrices in G. Then W is generated by the elements $s_i=(i,i+1)$ for $0\leq i\leq d+1$. The set W^μ of Konstant representatives for W/W_μ consists of:

$$w_0=1,w_1=s_1,w_2=s_2s_1,\ldots,w_d=s_ds_{d-1}\cdots s_1,\quad l_{[w_i]}=i.$$

We have $\Delta\setminus I_{[w_i]}=\{\alpha_{i+1},\ldots,\alpha_d\},|\Delta\setminus I_{[w_i]}|=d-i$. And $\rho_{[w_i]}(\mathbf{Z}/p^n)=\mathbf{Z}/p^n(-i)$, $\rho_{[w_i]}(\mathbf{Z}_p)=\mathbf{Z}_p(-i)$.

Set $\mathrm{Sp}_{d-i}(\mathbf{Z}/p^n):=v^G_{I_{[w_i]}}(\mathbf{Z}/p^n)$ and $\mathrm{Sp}^{\text{cont}}_{d-i}(\mathbf{Z}_p):=v^{G,\text{cont}}_{I_{[w_i]}}(\mathbf{Z}_p)$. Then our corollary follows from Theorem 6.4. □

We computed that:

$$H^i_{\text{ét},c}(\mathbb{H}^d_C,\mathbf{Z}/p^n)\simeq\mathrm{Sp}_{2d-i}(\mathbf{Z}/p^n)(d-i),\quad H^i_{\text{ét,c}}(\mathbb{H}^d_C,\mathbf{Z}_p)\simeq\mathrm{Sp}^{\text{cont}}_{2d-i}(\mathbf{Z}_p)(d-i).$$

In particular:

$$\begin{aligned}&H^i_{\text{ét},c}(\mathbb{H}^d_C,\mathbf{Z}/p^n)\simeq H^i_{\text{ét,c}}(\mathbb{H}^d_C,\mathbf{Z}_p)=0,\quad 0\leq i\leq d-1,\\&H^{2d}_{\text{ét},c}(\mathbb{H}^d_C,\mathbf{Z}/p^n)\simeq\mathbf{Z}/p^n(-d),\quad H^{2d}_{\text{ét,c}}(\mathbb{H}^d_C,\mathbf{Z}_p)\simeq\mathbf{Z}_p(-d).\end{aligned}$$

Recall that in [13, Th. 1.1] we have computed that:

$$H^i_{\text{ét}}(\mathbb{H}^d_C,\mathbf{Z}/p^n)\simeq\mathrm{Sp}_i(\mathbf{Z}/p^n)^*(-i),\quad H^i_{\text{ét}}(\mathbb{H}^d_C,\mathbf{Z}_p)\simeq\mathrm{Sp}^{\text{cont}}_i(\mathbf{Z}_p)^*(-i).$$

Hence (* denotes the linear dual):

Corollary 6.9. *There is a duality isomorphism of $\mathscr{G}_{\mathbf{Q}_p}\times G(\mathbf{Q}_p)$-modules*

$$H^i_{\text{ét}}(\mathbb{H}^d_C,\mathbf{Z}/p^n)(d)\simeq H^{2d-i}_{\text{ét,c}}(\mathbb{H}^d_C,\mathbf{Z}/p^n)^*,\quad H^i_{\text{ét}}(\mathbb{H}^d_C,\mathbf{Z}_p)(d)\simeq H^{2d-i}_{\text{ét,c}}(\mathbb{H}^d_C,\mathbf{Z}_p)^*.$$

6.2. **Orlik's fundamental complex.** In this section we will define a resolution of the constant sheaf $\mathbf{Z}/p^n$ on the complement of the period domain $\mathscr{F}^{\text{wa}}_C$ and compute the cohomology of its terms. The key geometric ideas are due to Orlik, see [51].

6.2.1. *Stratification of the constant sheaf.* Let $\mathrm{Sh}(Y_{C,\text{ét}})$ denote the topos of étale sheaves on Y_C. We refer the reader to the appendix for the review of the formalism of étale cohomology of pseudo-adic spaces developed by Huber in [36]. If X is an algebraic variety over C (for instance $Y_{I,C}$), let X^{ad} be the associated adic space over $\mathrm{Spa}(C, \mathscr{O}_C)$.

If Z is a closed pseudo-adic subspace of Y_C and $i : Z \to Y_C$ is the inclusion, define $F_Z := i_*i^*F \in \mathrm{Sh}(Y_{C,\text{ét}})$ for any sheaf $F \in \mathrm{Sh}(Y_{C,\text{ét}})$. Fix a subset $I \subset \Delta = \{\alpha_1, \dots, \alpha_d\}$. If T is a compact open subset of $X_I = J(\mathbf{Q}_p)/P_I(\mathbf{Q}_p)$, then

$$Z_{I,C}^T := \bigcup_{t\in T} tY_{I,C}^{\mathrm{ad}}$$

is a closed pseudo-adic subspace of Y_C thanks to the compactness of T (see [51, Lemma 3.2]) and for any $F \in \mathrm{Sh}(Y_{C,\text{ét}})$ we have a natural injection $F_{Z_{I,C}^T} \subset \prod_{t\in T} F_{tY_{I,C}^{\mathrm{ad}}}$. Therefore any partition $X_I = \coprod_{a\in A} T_a$ by (nonempty) compact open subsets[29] induces an embedding

$$\bigoplus_{a\in A} F_{Z_{I,C}^{T_a}} \subset \prod_{x\in X_I} F_{xY_{I,C}^{\mathrm{ad}}}.$$

Definition 6.10. If $F \in \mathrm{Sh}(Y_{C,\text{ét}})$, define $F_I \in \mathrm{Sh}(Y_{C,\text{ét}})$ as the *subsheaf of locally constant sections* of $\prod_{x\in X_I} F_{xY_{I,C}^{\mathrm{ad}}}$, i.e.,

$$F_I = \varinjlim_{c\in\mathscr{C}_I} F_c,$$

the limit being taken over the (pseudo-filtered) category $\mathscr{C}_I$ of compact open disjoint coverings of X_I ordered by refinement and F_c for $c = \{T_j\}_{j\in A} \in \mathscr{C}_I$ being the image of the natural embedding $\bigoplus_{j\in A} F_{Z_{I,C}^{T_j}} \hookrightarrow \prod_{x\in X_I} F_{xY_{I,C}^{\mathrm{ad}}}$.

The following computation will be essential for us later.

Proposition 6.11. (1) *If $\bar{x}$ is a geometric point of Y_C with support $x \in Y_C$, then, for all $F \in \mathrm{Sh}(Y_{C,\text{ét}})$, we have a natural isomorphism*

$$(F_I)_{\bar{x}} \simeq \mathrm{LC}(X_I(x), F_{\bar{x}}), \text{ where } X_I(x) = \{g \in X_I |\, x \in gY_{I,C}^{\mathrm{ad}}\}.$$

(2) *Let $i \in \mathbf{N}$. We have*

$$H^i_{\text{ét}}(Y_C, (\mathbf{Z}/p^n)_I) \simeq \mathrm{LC}(X_I, H^i_{\text{ét}}(Y_{I,C}, \mathbf{Z}/p^n)) \simeq i_{P_I}(\mathbf{Z}/p^n) \otimes H^i_{\text{ét}}(Y_{I,C}, \mathbf{Z}/p^n).$$

Proof. The first claim follows from the definition of F_I. Orlik's proof [51, Prop. 4.3] of the second claim for ℓ-adic sheaves goes through in our case, the key points of commuting with inductive and projective limits as well as the comparison algebraic-analytic being also valid p-adically. We describe briefly the essential steps of the proof. Since Y is quasi-compact and $\mathscr{C}_I$ is pseudo-filtered, [36, 2.3.13] yields an isomorphism

$$H^i_{\text{ét}}(Y_C, (\mathbf{Z}/p^n)_I) \simeq \varinjlim_{c\in\mathscr{C}_I} H^i_{\text{ét}}(Y_C, (\mathbf{Z}/p^n)_c) \simeq \varinjlim_{c\in\mathscr{C}_I} (\bigoplus_{T\in c} H^i_{\text{ét}}(Z_{I,C}^T, (\mathbf{Z}/p^n))).$$

We can write $Y_{I,C}^{\mathrm{ad}} = \bigcap_{s\in\mathbf{N}} Z_{I,C}^{T_s}$ [51, Lemma 4.4], for a family of compact open neighborhoods of the point $[P_I]$ of X_I such that $\cap_{s\in\mathbf{N}} T_s = [P_I]$. Hence, by [36, 2.4.6],

$$\varinjlim_{s\in\mathbf{N}} H^i_{\text{ét}}(Z_{I,C}^{T_s}, \mathbf{Z}/p^n) \simeq H^i_{\text{ét}}(Y_{I,C}^{\mathrm{ad}}, \mathbf{Z}/p^n) \simeq H^i_{\text{ét}}(Y_{I,C}, \mathbf{Z}/p^n),$$

the last isomorphism being a consequence of Huber's comparison theorem [36, Th. 3.7.2]. Combining the above we get

$$H^i_{\text{ét}}(Y_C, F_I) \simeq \varinjlim_{c\in\mathscr{C}_I} (\bigoplus_{T\in c} H^i_{\text{ét}}(Z_{I,C}^T, \mathbf{Z}/p^n)) \simeq \mathrm{LC}(X_I, H^i_{\text{ét}}(Y_{I,C}, \mathbf{Z}/p^n)),$$

---

[29] Note that A is necessarily finite since X_I is compact.

as desired. $\square$

6.2.2. *Acyclicity of the fundamental complex.* We will explain now how to create a complex out of the various F_I for $I \subset \Delta$. Choose an ordering on Δ and fix $F \in \mathrm{Sh}(Y_{C,\text{ét}})$. We will construct (following Orlik) maps $d_{I,I'} : F_{I'} \to F_I$ for all subsets I, I' of Δ with $|I'| - |I| = 1$, inducing the *fundamental complex*[30]:

$$C(F): \quad 0 \to F \to \bigoplus_{|\Delta\setminus I|=1} F_I \to \bigoplus_{|\Delta\setminus I|=2} F_I \to \cdots \to \bigoplus_{|\Delta\setminus I|=|\Delta|-1} F_I \to F_\varnothing \to 0. \tag{6.12}$$

We set $d_{I,I'} = 0$ when I is not a subset of I', so suppose that $I \subset I'$. We have a natural surjective map $p_{I,I'} : X_I \to X_{I'}$ (induced by $P_I \subset P_{I'}$), and for all $x \in X_{I'}, y \in X_I$ we have a natural map $F_{xY_{I'}} \to F_{yY_I}$, namely the zero map if $p_{I,I'}(y) \neq x$ and the map induced by the closed embedding $yY_I \to xY_{I'}$ otherwise. Unwinding the definitions of F_I and $F_{I'}$, we obtain a natural map $p_{I,I'} : F_{I'} \to F_I$, and we set $d_{I,I'} = (-1)^i p_{I,I'}$ if $I' = \{\alpha_1 < \cdots < \alpha_r\}$ and $I = I' \setminus \{\alpha_i\}$.

Recall that a sheaf F on $Y_{C,\text{ét}}$ is called *overconvergent* if the specialization morphism $F_{\xi_1} \to F_{\xi_2}$ is an isomorphism for any specialization of geometric points $\xi_2 \to \xi_1$. If F is an overconvergent sheaf, then it is easy to see that the terms of $C(F)$ are overconvergent [51, Lemma 3.4].

Theorem 6.13. *If F is an overconvergent sheaf on $Y_{C,\text{ét}}$, the fundamental complex $C(F)$ is acyclic.*

Proof. The arguments from the proofs of [51, Th. 3.3] and [53, Th. 2.1], for ℓ-adic sheaves, also work in this setting. We sketch them briefly. By overconvergence, it suffices to check the acyclicity of the stalk $C(F_\eta)$ of $C(F)$ at a maximal geometric point $\eta : \mathrm{Spa}(K, \mathscr{O}_K) \to \mathscr{F}_C$. If $x_\eta \in \mathscr{F}(K)$ is the induced point, the complex $C(F_\eta)$ is given, thanks to Proposition 6.11, by

$$0 \to F_{x_\eta} \to \bigoplus_{|\Delta\setminus I|=1} \mathrm{LC}(X_I(x_\eta), F_{x_\eta}) \to \cdots \to \bigoplus_{|\Delta\setminus I|=|\Delta|-1} \mathrm{LC}(X_I(x_\eta), F_{x_\eta}) \to \mathrm{LC}(X_\varnothing(x_\eta), F_{x_\eta}) \to 0 \tag{6.14}$$

where we recall that $X_I = J(\mathbf{Q}_p)/P_I(\mathbf{Q}_p)$, $X_I(x_\eta) = \{g \in X_I |\, x_\eta \in gY_I(K)\}$, and $P_\varnothing = P_0$. But this is a complex of locally constant functions (with values in F_{x_η}) on a subcomplex of the combinatoral building of J, whose simplices are given by

$$\{gP_Ig^{-1} | g \in J(\mathbf{Q}_p), x_\eta \in gY_I(K), I \subsetneq \Delta\}.$$

Its geometric realization (via the map τ from Section 5.3.2) is the subcomplex T_{x_η} of the spherical building $\mathscr{B}(J_{\text{der}})$ from Section 5.3.5. Since the complex T_{x_η} is contractible, by [62, Rem. 66], the complex (6.14) is acyclic. $\square$

6.3. **The key spectral sequence.** We will now evaluate the spectral sequence E_1 induced by the acyclic complex (6.12) for $F = \mathbf{Z}/p^n$:

$$E_1^{i,j} = H^j_{\text{ét}}(Y_C, \bigoplus_{|\Delta\setminus I|=i+1} (\mathbf{Z}/p^n)_I) \Rightarrow H^{i+j}_{\text{ét}}(Y_C, \mathbf{Z}/p^n).$$

In order to simplify some of the rather complicated formulae below we introduce the shorthands:

$$i_{I,n} := i_I(\mathbf{Z}/p^n),\ v_{I,n} := v_I(\mathbf{Z}/p^n),\ \rho_{[w],n} := \rho_{[w]}(\mathbf{Z}/p^n).$$

Lemma 6.15. *The above spectral sequence degenerates at E_2.*

Proof. By Proposition 6.11 we have

$$E_1^{i,j} \simeq \bigoplus_{|\Delta\setminus I|=i+1} \mathrm{LC}(X_I, H^j_{\text{ét}}(Y_{I,C}, \mathbf{Z}/p^n)) \simeq \bigoplus_{|\Delta\setminus I|=i+1} i_{I,n} \otimes H^j_{\text{ét}}(Y_{I,C}, \mathbf{Z}/p^n).$$

Define

$$I_{[w]} = \{\alpha \in \Delta |\ \ \mathscr{P}(w\mu - \nu, \omega_\alpha) \leq 0\}.$$

---

[30] Check [17, Ch. XI] for a "geometric" construction of this complex.

It is the minimal subset of Δ such that $[w]\in\Omega_{I_{[w]}}$, thus we have $[\omega]\in\Omega_I$ if and only if $I_{[w]}\subset I$. From the Bruhat decomposition of $H^j_{\text{ét}}(Y_{I,C},\mathbf{Z}/p^n)$ obtained in Corollary 5.11, using Remark 5.14, we obtain a decomposition of the spectral sequence E_1:

$$E_1=\bigoplus_{[w]\in W^\mu/\mathscr{G}_{E_s}} E_{1,[w]},$$

where $E_{1,[w]}$ is the complex living just in row $2l_{[w]}$:

$$E_{1,[w]}=(\bigoplus_{\substack{I_{[w]}\subset I\\ |\Delta\setminus I|=1}} i_{I,n}\otimes\rho_{[w],n}\to\bigoplus_{\substack{I_{[w]}\subset I\\ |\Delta\setminus I|=2}} i_{I,n}\otimes\rho_{[w],n}\to\cdots\to i_{I_{[w]},n}\otimes\rho_{[w],n})[-2l_{[w]}].$$

We get an exact sequence of complexes[31]

$$0\to i_{\Delta,n}\otimes\rho_{[w],n}[-2l_{[w]}+1]\to C_{I_{[w]}}(\mathbf{Z}/p^n)\otimes\rho_{[w],n}[-2l_{[w]}]\to E_{1,[w]}\to 0.$$

By Proposition 6.1 and noting that $\rho_{[w],n}$ is a free $\mathbf{Z}/p^n$-module, the only nonzero terms of $E_{2,[w]}$ are given as follows:

$$E^{0,2l_{[w]}}_{2,[w]}\simeq\begin{cases} i_{I_{[w]},n}\otimes\rho_{[w],n} & \text{if } |\Delta\setminus I_{[w]}|=1,\\ i_{\Delta,n}\otimes\rho_{[w],n},\quad E^{i,2l_{[w]}}_{2,[w]}\simeq v_{I_{[w]},n}\otimes\rho_{[w],n},\quad i=|\Delta\setminus I_{[w]}|-1 & \text{if } |\Delta\setminus I_{[w]}|>1.\end{cases}$$

To see that $E_2=E_\infty$ note that the nontrivial differentials on E_i, $i\geq 2$, can only go from $E_{i,[w]}$ to $E_{i,[w']}$ for $[w]\neq[w']$, $l_{[w]}\neq l_{[w']}$. But such maps have to be trivial by a weight argument. Indeed, all the terms of E_2 are free modules over $\mathbf{Z}/p^n$ and are given by the tensor product of a $J(\mathbf{Q}_p)$-module and a $\mathscr{G}_{E_s}$-module. We also have $E_2(\mathbf{Z}/p^n)\otimes_{\mathbf{Z}/p^n}\mathbf{Z}/p^m\simeq E_2(\mathbf{Z}/p^m)$ for any integer $n>m$. Thus the nontrivial differentials of the spectral sequence E_i, $i\geq 2$, would yield nontrivial maps between projective systems $\{\mathbf{Z}/p^n(a)\}_n$ and $\{\mathbf{Z}/p^n(b)\}_n$, $a\neq b$. And this is not possible (see [51, Sec. 5] for details). ☐

The next proposition crucially uses the results of the first section.

Proposition 6.16. *We have*

$$\begin{aligned}H^*_{\text{ét}}(Y_C,\mathbf{Z}/p^n)\simeq&\Big(\bigoplus_{|\Delta\setminus I_{[w]}|=1} i_{I_{[w]},n}\otimes\rho_{[w],n}[-2l_{[w]}]\Big)\bigoplus\\ &\bigoplus_{|\Delta\setminus I_{[w]}|>1}\big((i_{\Delta,n}\otimes\rho_{[w],n}[-2l_{[w]}])\oplus(v_{I_{[w]},n}\otimes\rho_{[w],n}[-2l_{[w]}-|\Delta\setminus I_{[w]}|+1])\big).\end{aligned}$$

Proof. Fix $i,j\in\mathbf{N}$. Using Lemma 6.15 and its proof we can compute the grading induced by the spectral sequence E_1:

$$\begin{aligned}\operatorname{gr}^i(H^j_{\text{ét}}(Y_C,\mathbf{Z}/p^n))=E^{i,j-i}_\infty=E^{i,j-i}_2&\simeq\bigoplus_{[w]\in W^\mu/\mathscr{G}_{E_s}} E^{i,j-i}_{2,[w]}\\ &\simeq\begin{cases}\big(\bigoplus_{[w]\in T_1} i_{I_{[w]},n}\otimes\rho_{[w],n}\big)\bigoplus\big(\bigoplus_{[w]\in T_2} i_{\Delta,n}\otimes\rho_{[w],n}\big) & \text{if } i=0,\\ \bigoplus_{[w]\in T_3} v_{I_{[w]},n}\otimes\rho_{[w],n} & \text{if } i>0,\end{cases}\end{aligned}$$

where we set

$$T_1:=\{[w]\in W^\mu/\mathscr{G}_{E_s}\,|\,|\Delta\setminus I_{[w]}|=1,2l_{[w]}=j\},\quad T_2:=\{[w]\in W^\mu/\mathscr{G}_{E_s}\,|\,|\Delta\setminus I_{[w]}|>1,2l_{[w]}=j\},$$
$$T_3:=\{[w]\in W^\mu/\mathscr{G}_{E_s}\,|\,2l_{[w]}+|\Delta\setminus I_{[w]}|-1=j,i=|\Delta\setminus I_{[w]}|-1\}.$$

It suffices to show that this grading splits. Fix j. We start by proving the following

Lemma 6.17. *The cohomology groups $H^i_{\text{ét}}(Y_C,\mathbf{Z}/p^n)$ are smooth $J(\mathbf{Q}_p)$-modules.*

---

[31] Recall that the complexes $C_I(\mathbf{Z}/p^n)$ are defined in Proposition 6.1.

Proof. We start by observing that Y_C is proper, being quasi-compact and closed in $\mathscr{F}_C$, which is proper (see [36, Lemma 5.3.3]). It follows that $\mathrm{R}\Gamma_{\text{ét},c}(Y_C,\mathbf{Z}/p^n)\simeq\mathrm{R}\Gamma_{\text{ét}}(Y_C,\mathbf{Z}/p^n)$ and the distinguished triangle associated to the triple $(\mathscr{F}^{\mathrm{wa}},\mathscr{F},Y)$ becomes

$$\mathrm{R}\Gamma_{\text{ét},c}(\mathscr{F}^{\mathrm{wa}}_C,\mathbf{Z}/p^n)\longrightarrow\mathrm{R}\Gamma_{\text{ét}}(\mathscr{F}_C,\mathbf{Z}/p^n)\xrightarrow{i^*}\mathrm{R}\Gamma_{\text{ét}}(Y_C,\mathbf{Z}/p^n). \tag{6.18}$$

Consider the induced long exact sequence of cohomology groups. By a result of Berkovich [6, Cor. 7.8], we know that $H^i_{\text{ét},c}(\mathscr{F}^{\mathrm{wa}}_C,\mathbf{Z}/p^n)$ is a smooth $J(\mathbf{Q}_p)$-module. We note that one cannot apply this result directly to Y, which is only a pseudo-adic space. And, of course, $H^i_{\text{ét}}(\mathscr{F}_C,\mathbf{Z}/p^n)$ is a smooth representation of $J(\mathbf{Q}_p)$, of finite type over $\mathbf{Z}/p^n$. Our lemma follows then by induction on i, using the next lemma. $\square$

Lemma 6.19. *Let G be a p-adic analytic group and let π be a representation of G over $\mathbf{Z}/p^n$ living in an exact sequence*

$$0\to\sigma\to\pi\to\tau\to 0,$$

with σ and τ smooth representations of G over $\mathbf{Z}/p^n$ and σ finitely generated as $\mathbf{Z}/p^n$-module. Then π is a smooth representation of G.

Proof. Since σ is finitely generated over $\mathbf{Z}/p^n$ and smooth, there is an open subgroup H of G acting trivially on σ. Replacing G by H we may thus assume that G acts trivially on σ. Let $v\in\pi$. Since τ is smooth, there is an open subgroup K of G fixing the image of v in τ. Shrinking K, we may assume that K is a uniform pro-p group. Since $kv-v\in\sigma$, for $k\in K$, we have $(g-1)(k-1)v=0$ for $g\in G$ and $k\in K$. Using the binomial formula and the fact that p^n kills π, it follows that $k^{p^n}v=v$ for $k\in K$. Since K^{p^n} is open in K and thus in G, we are done. $\square$

To show that the filtration for $i>0$ splits, consider the equation

$$2l_{[w]}+|\Delta\setminus I_{[w]}|-1=j=2l_{[w']}+|\Delta\setminus I_{[w']}|-1$$

with $[w],[w']\in W^\mu/\mathscr{G}_{E_s}$. If $l_{[w]}\neq l_{[w']}$ this equation implies that $|\Delta\setminus I_{[w]}|$ and $|\Delta\setminus I_{[w']}|$ differ by at least two. Hence $|I_{[w]}|$ and $|I_{[w']}|$ differ by at least two as well. Since $H^j_{\text{ét}}(Y,\mathbf{Z}/p^n)$ are smooth $J(\mathbf{Q}_p)$-modules and, by Theorem 1.8, the Ext group $\mathrm{Ext}^1_{J(\mathbf{Q}_p)}(v_{I_{[w]}},v_{I_{[w']}})$ in the category of smooth representations is trivial, we have a splitting of $J(\mathbf{Q}_p)$-modules. This splitting is automatically Galois equivariant: call the section s and consider gs, $g\in\mathscr{G}_{E_s}$. The map $t:=s-gs$ decomposes into a direct sum of maps between generalized Steinberg representations v_I and $v_{I'}$ with I and I' differing by at least two elements. Hence, by Proposition 2.1, all these maps are trivial and thus $t=0$, as wanted.

To include $i=0$, we start by showing that

$$\begin{aligned}&\mathrm{Ext}^1_{J(\mathbf{Q}_p)}(i_{I_{[w]},n},v_{I_{[w'],n}})=0,\quad w\in T_1,w'\in T_3,\\&\mathrm{Ext}^1_{J(\mathbf{Q}_p)}(i_{\Delta,n},v_{I_{[w'],n}})=0,\quad w'\in T_3.\end{aligned}\tag{6.20}$$

For the first equality, consider the exact sequence (recall that $|\Delta\setminus I_{[w]}|=1$)

$$0\to i_{\Delta,n}\to i_{I_{[w]},n}\to v_{I_{[w]},n}\to 0 \tag{6.21}$$

It yields the exact sequence

$$\mathrm{Ext}^1_{J(\mathbf{Q}_p)}(v_{I_{[w]},n},v_{I_{[w'],n}})\to\mathrm{Ext}^1_{J(\mathbf{Q}_p)}(i_{I_{[w]},n},v_{I_{[w'],n}})\to\mathrm{Ext}^1_{J(\mathbf{Q}_p)}(i_{\Delta,n},v_{I_{[w'],n}})$$

Since

$$|\Delta\setminus I_{[w']}|=-2l_{[w']}+1+j=-2l_{[w']}+2l_{[w]}+1\geq 3$$

and $|\Delta\setminus I_{[w]}|=1$, $|I_{[w]}|$ and $|I_{[w']}|$ differ by at least two and the first term in the above sequence is zero by Theorem 1.8; similarly, since $|\Delta|$ and $|I_{[w']}|$ differ by at least two, the right term of the above sequence is zero by Theorem 1.8. Hence we have obtained the first equality in (6.20) and along the way we have shown the second equality in (6.20) as well.

Moreover, the $J(\mathbf{Q}_p)$-equivariant sections are automatically Galois equivariant: one argues as above using in addition the exact sequence

$$\mathrm{Hom}_{J(\mathbf{Q}_p)}(v_{I_{[w]},n}, v_{I_{[w']},n}) \to \mathrm{Hom}_{J(\mathbf{Q}_p)}(i_{I_{[w]},n}, v_{I_{[w']},n}) \to \mathrm{Hom}_{J(\mathbf{Q}_p)}(i_{\Delta,n}, v_{I_{[w']},n}),$$

induced by the exact sequence (6.21), which shows that the middle term is trivial since so are the other two terms (by Proposition 2.1). $\square$

6.4. **End of the proof of Theorem 6.4.** (1) *Torsion compactly supported étale cohomology.* In order to prove claim (1), we use the distinguished triangle (6.18). An argument based on the Bruhat decomposition of $\mathscr{F} \simeq G_{E_s}/P(\mu)$, as in the proof of Corollary 5.11, shows that

$$\mathrm{R}\Gamma_{\text{ét}}(\mathscr{F}_C, \mathbf{Z}/p^n) \simeq \bigoplus_{[w]\in W^\mu/\mathscr{G}_{E_s}} \rho_{[w],n}(\mathbf{Z}/p^n)[-2l_{[w]}] := \widetilde{E}_1$$

The map

$$i^* : \mathrm{R}\Gamma_{\text{ét}}(\mathscr{F}_C, \mathbf{Z}/p^n) \to \mathrm{R}\Gamma_{\text{ét}}(Y_C, \mathbf{Z}/p^n)$$

can be represented by the map of complexes $\widetilde{E}_1 \to E_1$ induced by the canonical maps

$$\rho_{[w],n} = \iota_{\Delta,n} \otimes \rho_{[w],n} \to \iota_{I,n} \otimes \rho_{[w],n}, \quad I_{[w]} \subset I, |\Delta \setminus I| = 1.$$

By Propositions 6.1 and 6.16,

$$\mathrm{Cone}(i^*)[-1] \simeq \bigoplus_{[w]\in W^\mu/\mathscr{G}_{E_s}} v_{I_{[w]},n} \otimes \rho_{[w],n}[-2l_{[w]} - |\Delta \setminus I_{[w]}|],$$

as wanted.

(2) *p-adic compactly supported étale cohomology.* For claim (2), take the exact sequence (see Section A.2.2):

$$0 \to \mathrm{R}^1 \varprojlim_n H^{i-1}_{\text{ét},c}(\mathscr{F}^{\text{wa}}_C, \mathbf{Z}/p^n) \to H^i_{\text{ét},c}(\mathscr{F}^{\text{wa}}_C, \mathbf{Z}_p) \to \varprojlim_n H^i_{\text{ét},c}(\mathscr{F}^{\text{wa}}_C, \mathbf{Z}/p^n) \to 0.$$

Since

$$v_{I_{[w]},n} \simeq v_{I_{[w]},n+1} \otimes_{\mathbf{Z}/p^{n+1}} \mathbf{Z}/p^n, \quad v^{\text{cont}}_{I_{[w]}}(\mathbf{Z}_p) \simeq \varprojlim_n v_{I_{[w]},n},$$

the pro-system $\{H^{i-1}_{\text{ét},c}(\mathscr{F}^{\text{wa}}_C, \mathbf{Z}/p^n)\}_n$ is Mittag-Leffler hence the above exact sequence and claim (1) yield the isomorphism

$$H^i_{\text{ét},c}(\mathscr{F}^{\text{wa}}_C, \mathbf{Z}_p) \xrightarrow{\sim} \varprojlim_n H^i_{\text{ét},c}(\mathscr{F}^{\text{wa}}_C, \mathbf{Z}/p^n)$$

and claim (2).

Remark 6.22. In [53] Orlik computed ℓ-adic compactly supported étale cohomology for $\ell \neq p$. In the context of Theorem 6.4 and for ℓ sufficiently generic with respect to G he obtained an isomorphism of $\mathscr{G}_{E_s} \times J(\mathbf{Q}_p)$-modules

$$H^*_{\text{ét,c,Hu}}(\mathscr{F}^{\text{wa}}_C, \mathbf{Z}_\ell) \simeq \bigoplus_{[w]\in W^\mu/\mathscr{G}_{E_s}} v^J_{I_{[w]}}(\mathbf{Z}_\ell) \otimes \rho_{[w]}(\mathbf{Z}_\ell)[-n_{[w]}], \tag{6.23}$$

where $H^*_{\text{ét,c,Hu}}$ denotes Huber's compactly supported cohomology. The proof follows the proof for torsion coefficients with two main differences:

1. It starts with the distinguished triangle associated to the triple $(\mathscr{F}^{\text{wa}}, \mathscr{F}, Y)$:
$$\mathrm{R}\Gamma_{\text{ét,c,Hu}}(\mathscr{F}^{\text{wa}}_C, \mathbf{Z}_p) \longrightarrow \mathrm{R}\Gamma_{\text{ét}}(\mathscr{F}_C, \mathbf{Z}_p) \xrightarrow{i^*} \mathrm{R}\Gamma_{\text{ét}}(Y_C, \Lambda), \quad \Lambda = i^*\mathrm{R}\pi_*(\mathbf{Z}/p^n)_n.$$
2. It uses the fact (from [16, Appendix B.2]) that the representations $H^*_{\text{ét,c,Hu}}(\mathscr{F}^{\text{wa}}_C, \mathbf{Z}_p), \mathrm{R}\Gamma_{\text{ét}}(Y_C, \Lambda), \ldots$ are smooth $J(\mathbf{Q}_p)$-modules.

As we have mentioned in the introduction the p-adic analog of the isomorphism (6.23) is false and the smoothness property mentioned above does not hold.

Appendix A. Adic potpourri

We gather here, as a reference, some basic facts concerning pseudo-adic spaces and compactly supported étale cohomology.

A.1. **Pseudo-adic spaces.**

We start with pseudo-adic spaces. Recall that, Huber defines in [36] the category PPA of pre-pseudo-adic spaces, consisting of pairs $X = (\underline{X}, |X|)$, where $\underline{X}$ is an adic space and $|X|$ a subset of $\underline{X}$, morphisms $X \to Y$ being morphisms of adic spaces $\underline{X} \to \underline{Y}$ that send $|X|$ into $|Y|$. A morphism $f : X \to Y$ induces therefore a morphism of adic spaces $\underline{f} : \underline{X} \to \underline{Y}$ and a map of topological spaces $|f| : |X| \to |Y|$ (we endow $|X|$ with the topology induced from $\underline{X}$). We say that f is *étale* if $\underline{f}$ is étale and if $|X|$ is open in $\underline{f}^{-1}(|Y|)$ (this implies that $|f|$ is an open map). The étale site $X_{\text{ét}}$ of X is the category of pre-pseudo-adic spaces Y étale over X with the topology such that a family of morphisms $f_i : Y_i \to Y$ in this category is a covering if $|Y| = \cup_i |f_i|(|Y_i|)$.

We mention the following properties of this construction, which we need, and we refer the reader to Huber's book [36] for the proofs and details (see especially Sections 1.10, 2.3):

1. The category PPA contains (as full subcategory) the category of adic spaces (via $X \mapsto (X, |X|)$) and the étale topoi of X and $(X, |X|)$ are equivalent.
2. If X is an adic space and $S \subset T$ are subsets of X, the natural morphism $i : (X, S) \to (X, T)$ in PPA satisfies $i^* i_* F \simeq F$, for all $F \in \mathrm{Sh}((X, S)_{\text{ét}})$, thus $i_* : \mathrm{Sh}((X, S)_{\text{ét}}) \to \mathrm{Sh}((X, T)_{\text{ét}})$ is fully faithful. Moreover, if S is closed in T, then i_* is exact and identifies $\mathrm{Sh}((X, S)_{\text{ét}})$ with the full subcategory of $\mathrm{Sh}((X, T)_{\text{ét}})$ consisting of sheaves F whose restriction to $(T - S)_{\text{ét}}$ is the final object of $\mathrm{Sh}((T - S)_{\text{ét}})$ ([36, Lemma 2.3.11]).
3. Let PA be the full subcategory of PPA consisting of *pseudo-adic spaces*, i.e., those X for which $|X|$ is convex and locally pro-constructible in $\underline{X}$. An object X of PA is called quasi-compact/quasi-separated if $|X|$ is so, and a map $f : X \to Y$ in PA is called quasi-compact/quasi-separated if $|f|$ is so. If $f : X \to Y$ is a quasi-compact quasi-separated morphism in PA and if f is adic (i.e., $\underline{f}$ is adic), then $\mathrm{R}^n f_*$ commutes with pseudo-filtered inductive limits. If $X \in PA$ is quasi-compact quasi-separated, then $H^n_{\text{ét}}(X, -)$ commutes with pseudo-filtered inductive limits. ([36, Lemma 2.3.13]).
4. If x is a point of an adic space X and if K is the henselization of the residue class field $k(x)$ with respect to the valuation ring $k(x)^+$, there is a natural equivalence of categories $\mathrm{Sh}((X, \{x\})_{\text{ét}}) \simeq \mathrm{Sh}(\mathrm{Spec}(K)_{\text{ét}})$ ([36, Prop. 2.3.10]).
5. Let P be one of the properties "open, closed, locally closed". A P-subspace of $X \in PPA$ is an object $Y \in PPA$ for which $\underline{Y}$ is a P-subspace of $\underline{X}$ and $|Y|$ is a P-subspace of $|X|$. The notion of P-embedding in PPA is defined in the obvious way. If $i : X \to Y$ is a locally closed embedding in PPA then i induces an equivalence $\mathrm{Sh}(X_{\text{ét}}) \simeq \mathrm{Sh}((\underline{Y}, i(|X|))_{\text{ét}})$ ([36, Cor. 2.3.8]). In particular if $i : X \to Y$ is a locally closed embedding of adic spaces then $\mathrm{Sh}(X_{\text{ét}}) \simeq \mathrm{Sh}((Y, i(|X|))_{\text{ét}})$.
6. A morphism $f : X \to Y$ in PPA is finite if $\underline{f}$ is finite and $|X|$ is closed in $\underline{f}^{-1}(|Y|)$. If $f : X \to Y$ is a finite morphism in PA, then $f_* : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}})$ is exact and commutes with any base change in PA $Y' \to Y$ ([36, Prop. 2.6.3]).
7. A *geometric point* (in the category PPA) is an object $S \in PA$ such that $\underline{S}$ is the adic spectrum of a separably algebraically closed affinoid field ([36, 1.1.5]) and $|S| = \{s\}$, where s is the closed point of $\underline{S}$ ([36, 1.1.6]). For a geometric point S, the functor $\Gamma(S, -)$ induces an equivalence $\mathrm{Sh}(S_{\text{ét}}) \simeq$ Sets. A *geometric point of* $X \in PPA$ is a morphism $u : S \to X$ in PPA, where S is a geometric point. The stalk of $F \in \mathrm{Sh}(X_{\text{ét}})$ at S is then $F_S = \Gamma(S, u^* F)$. Somewhat more explicitly, $F_S \simeq \varinjlim_{(V,v)} F(V)$, the limit being over the cofiltered category C_S of pairs (V, v), where V is étale over X and $v : S \to V$ is an X-morphism.

   The support of u is by definition $u(|S|) \in |X|$. Two geometric points with the same support yield isomorphic stalk functors. Moreover, each $x \in X$ induces a geometric point

$\bar{x}$ of X with support x and the family of functors $\mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sets}, F \to F_{\bar{x}}$, for $x \in |X|$, is conservative ([36, 2.5.5]). If $f : X \to Y$ is a morphism of analytic pseudo-adic spaces (i.e., $\underline{X}, \underline{Y}$ are analytic adic spaces) and f is of weakly finite type and quasi-separated, then, for any maximal point y of $|Y|$ and any $F \in \mathrm{Sh}(X_{\text{ét}})$, we have a natural isomorphism ([36, Th. 2.6.2])

$$(\mathrm{R}^n f_* F)_{\bar{y}} \simeq H^n_{\text{ét}}(X \times_Y \bar{y}, F).$$

(8) One can define (see [36, Sec. 2.5]), for each geometric point ξ of $X \in PPA$, the *strict localization* $X(\xi)$ *of* X *at* ξ. It comes with an X-morphism $\xi \to X(\xi)$, and the isomorphism class of $X(\xi)$ as X-space depends only on the support of ξ. If $X \in PA$ and ξ, ξ' are geometric points of X, a *specialization morphism* $u : \xi \to \xi'$ is an X-morphism in PA $X(\xi) \to X(\xi')$. It induces functorial maps $u^*(F) : F_{\xi'} \to F_\xi$ for $F \in \mathrm{Sh}(X_{\text{ét}})$, via the natural isomorphisms $\Gamma(X(\xi), F|X(\xi)) \simeq F_\xi$ and $\Gamma(X(\xi'), F|X(\xi')) \simeq F_{\xi'}$.

A.2. **Compactly supported cohomology.** We survey Huber's compactly supported étale cohomology and introduce continuous compactly supported étale cohomology.

A.2.1. *Huber's compactly supported étale cohomology.* Huber defined compactly supported étale cohomology of analytic pseudo-adic spaces in [36, Ch. 5]; in [37] he extended this definition to ℓ-adic sheaves. We will briefly recall its properties.

Fix a prime ℓ. Let X be a taut separated pseudo-adic space locally of $^+$weakly finite type over C (i.e., over $\mathrm{Spa}(C, \mathscr{O}_C)$). For $i \geq 0$, we set

$$H^i_{\text{ét,c,Hu}}(X, \mathbf{Z}_\ell) := H^i \mathrm{R}\Gamma_{\mathrm{c,Hu}}(X_{\text{ét}}, (\mathbf{Z}/\ell^n)_n), \quad H^i_{\text{ét,c,Hu}}(X, \mathbf{Q}_\ell) := H^i_{\text{ét,c,Hu}}(X, \mathbf{Z}_\ell) \otimes \mathbf{Q}_\ell,$$

where the functor $\mathrm{R}\Gamma_{\mathrm{c,Hu}}$ is defined in the following way.

If X is partially proper, then it is the right derived functor of $\Gamma_{\mathrm{c,Hu}}$, i.e., of the left exact functor

$$\Gamma_{\mathrm{c,Hu}} : \mathrm{mod}(X_{\text{ét}} - \mathbf{Z}_\ell^\bullet) \to \mathrm{mod}(\mathbf{Z}_\ell), \quad (F_n)_n \mapsto \Gamma_c(X_{\text{ét}}, \varprojlim_n F_n).$$

Here $\mathrm{mod}(X_{\text{ét}} - \mathbf{Z}_\ell^\bullet)$ is the category of projective systems $(F_n)_n$ of $\mathbf{Z}_\ell$-modules on $X_{\text{ét}}$ such that $p^n F_n = 0$, $n \in \mathbf{N}$. Recall that, for an étale sheaf F, $\Gamma_c(X_{\text{ét}}, F)$ denotes the abelian group of global sections whose support is proper.

In general one sets

$$\mathrm{R}\Gamma_{\mathrm{c,Hu}}(X_{\text{ét}}, (F_n)_n) := \mathrm{R}\Gamma_{\mathrm{c,Hu}}(\overline{X}_{\text{ét}}, (i_! F_n)_n),$$

where $i : X \hookrightarrow \overline{X}$ is a locally closed embedding and $\overline{X}$ is partially proper. This definition is, of course, independent of the chosen partially proper compactification. We have

$$\Gamma_{\mathrm{c,Hu}}(X_{\text{ét}}, (F_n)_n) = \{(s_n)_n \in \varprojlim_n \Gamma(X_{\text{ét}}, F_n) | \overline{\cup_n \mathrm{supp}(s_n)} \text{ is proper}\}.$$

We list the following properties:

(1) If X is proper then

$$\mathrm{R}\Gamma_{\mathrm{c,Hu}}(X_{\text{ét}}, (F_n)_n) \simeq \mathrm{R}\Gamma(X_{\text{ét}}, (F_n)_n).$$

(2) An isomorphism [37, Lemma 2.3] of exact functors from $D^+(\mathrm{mod}(X_{\text{ét}} - \mathbf{Z}_\ell^\bullet))$ to $D^+(\mathrm{mod}(\mathbf{Z}_\ell))$:

$$\mathrm{R}\Gamma_{\mathrm{c,Hu}} = \mathrm{R}\Gamma_! \circ \mathrm{R}\pi_*,$$

for the *discretization* functor

$$\pi_* : \mathrm{mod}(X_{\text{ét}} - \mathbf{Z}_\ell^\bullet) \to \mathrm{mod}(X_{\text{ét}} - \mathbf{Z}_\ell), \quad (F_n)_n \mapsto \varprojlim_n F_n,$$

and the functor

$$\Gamma_! : \mathrm{mod}(X_{\text{ét}} - \mathbf{Z}_\ell) \to \mathrm{mod}(\mathbf{Z}_\ell), \quad F \mapsto \Gamma_c(X_{\text{ét}}, F).$$

(3) If X is quasi-compact, there is an exact sequence [37, Cor. 2.4]
$$0 \to \mathrm{R}^1 \varprojlim_n H^{i-1}_{\text{ét},c}(X, F_n) \to H^i_{\text{ét,c,Hu}}(X, (F_n)_n) \to \varprojlim_n H^i_{\text{ét},c}(X, F_n) \to 0$$

(4) Let U be a taut open subspace of X, let $Z = X \setminus U$, and let $i : Z \hookrightarrow X$ be the inclusion. Assume that X, U are partially proper. Then we have a distinguished triangle
$$\mathrm{R}\Gamma_{\mathrm{c,Hu}}(U_{\text{ét}}, (F_n|U)_n) \to \mathrm{R}\Gamma_{\mathrm{c,Hu}}(X_{\text{ét}}, (F_n)_n) \to \mathrm{R}\Gamma_c(Z_{\text{ét}}, i^*\mathrm{R}\pi_*(F_n)_n) \tag{A.1}$$

(5) Let $\mathbb{U}$ be an open covering of X such that every $U \in \mathbb{U}$ is taut and, for every $U, V \in \mathbb{U}$, there exists a $W \in \mathbb{U}$ such that $U \cup V \subset W$. Then, the map
$$\varinjlim_{U \in \mathbb{U}} H^i_{\text{ét,c,Hu}}(U, (F_n|U)_n) \to H^i_{\text{ét,c,Hu}}(X, (F_n)_n), \quad i \geq 0,$$
is an isomorphism [37, Prop. 2.1.].

(6) Let X be adic and partially proper and let G be a locally profinite group acting continuously on X. Then $H^i_{\text{ét},c}(X, \mathbf{Z}/\ell^n)$, $i \geq 0$, is a smooth G-module [6, Cor. 7.8].

Let us now distinguish two cases.

(i) **The case $\ell \neq p$.** We can say more in this case.

(1) If X is as at the beginning of this section and of finite type over C and if $(F_n)_n$ is a quasi-constructible $\mathbf{Z}_\ell^\bullet$-module on $X_{\text{ét}}$ then the natural map
$$H^i_{\text{ét,c,Hu}}(X, (F_n)_n) \to \varprojlim_n H^i_{\text{ét},c}(X, F_n)$$
is a bijection. Moreover, the projective system $(H^i_{\text{ét},c}(X, F_n))_n$ is ℓ-adic and every $H^i_{\text{ét},c}(X, F_n)$ is a finitely generated $\mathbf{Z}_\ell$-module (hence also $H^i_{\text{ét,c,Hu}}(X, (F_n)_n)$ is a finitely generated $\mathbf{Z}_\ell$-module).

(2) If $X = Y^{\mathrm{ad}}$, for a separated scheme Y of finite type over C, and if $(F_n)_n$ is a constructible $\mathbf{Z}_\ell^\bullet$-module on $Y_{\text{ét}}$, there is a natural isomorphism
$$H^i_{\text{ét},c}(Y, (F_n)_n) \xrightarrow{\sim} H^i_{\text{ét,c,Hu}}(X, (F_n)_n).$$

(3) Let X be adic and partially proper and let G be a locally profinite group, with an open pro-p subgroup, acting continuously on X. Let $(F_n)_n$ be a locally constant overconvergent $\mathbf{Z}_\ell^\bullet$-module equipped with a compatible discrete G-action (see [16, B.1.3] for the definition). Then $H^i_{\text{ét,c,Hu}}(X, (F_n)_n)$, $i \geq 0$, is a smooth G-module [16, Prop. B.2.5], [22, 4.1.19].

(ii) **The case $\ell = p$.** In this case cohomology with compact support behaves very differently. We will discuss an example.

Example A.2. Let $\mathbb{A}^1_C$ be the adic affine space of dimension 1; this is a period domain, with $G := \mathbb{G}_{m,\mathbf{Q}_p} \times \mathbb{G}_{m,\mathbf{Q}_p}$ the relevant reductive group [3, 5.3.1, 4.2.2]. We have the exact sequence
$$0 \to H^1_{\text{ét}}(x_\infty, i_\infty^*\mathrm{R}\pi_*(\mathbf{Z}/p^n(1))_n) \to H^2_{\text{ét,c,Hu}}(\mathbb{A}^1_C, \mathbf{Z}_p(1)) \to H^2_{\text{ét}}(\mathbb{P}^1_C, \mathbf{Z}_p(1)) \to H^2_{\text{ét}}(x_\infty, i_\infty^*\mathrm{R}\pi_*(\mathbf{Z}/p^n(1))_n),$$
where $i_\infty : x_\infty \hookrightarrow \mathbb{P}^1_C$ is the point at infinity. Picking the fundamental neighbourhoods of x_∞ consisting of closed balls, we compute easily that
$$i_\infty^*\mathrm{R}^i\pi_*(\mathbf{Z}/p^n(1))_n \simeq \begin{cases} \mathbf{Z}_p(1) & \text{if } i = 0, \\ \varinjlim_j (\varprojlim_n H^1_{\text{ét}}(E(j), \mathbf{Z}/p^n(1))) & \text{if } i = 1, \\ 0 & \text{if } i \geq 2, \end{cases}$$
where $E(j)$ is the closed ball centered at x_∞ and of radius p^{-j}. We used here the fact that $H^i_{\text{ét}}(E(j), \mathbf{Z}/p^n(1)) = 0, i \geq 2$. Since $\mathrm{Pic}(E(j)) = 0$, the Kummer exact sequence implies that
$$H^1_{\text{ét}}(E(j), \mathbf{Z}/p^n(1)) \simeq C\{T^{-1}\}^*/C\{T^{-1}\}^{*p^n}.$$

Hence $H^2_{\text{ét}}(x_\infty, i_\infty^*\mathrm{R}\pi_*(\mathbf{Z}/p^n(1))_n) = 0$ and $H^1_{\text{ét}}(x_\infty, i_\infty^*\mathrm{R}\pi_*(\mathbf{Z}/p^n(1))_n) \simeq \varinjlim_n C\{(p^nT)^{-1}\}^{*\wedge}$. It follows that

$$H^2_{\text{ét,c,Hu}}(\mathbb{A}^1_C, \mathbf{Z}_p(1)) \simeq (\varinjlim_n C\{(p^nT)^{-1}\}^{*\wedge}) \oplus \mathbf{Z}_p. \tag{A.3}$$

In the case $\ell \neq p$, the same computation gives $\mathbf{Z}_\ell$ as a result since $C\{(p^nT)^{-1}\}^{*\wedge}$ is ℓ-divisible. Note that $C\{T^{-1}\}^*/C^* = 1 + T^{-1}\mathfrak{m}_C\{T^{-1}\}$ and that its image by the logarithm satisfies

$$pT^{-1}\mathfrak{m}_C\{T^{-1}\} \subset \log\big(1 + T^{-1}\mathfrak{m}_C\{T^{-1}\}\big) \subset (pT)^{-1}\mathfrak{m}_C\{(pT)^{-1}\}.$$

One gets the same inclusions for the p-adic completion. Hence the above inductive limit is isomorphic, via the logarithm, to the inductive limit of the $(p^nT)^{-1}\mathfrak{m}_C\{(p^nT)^{-1}\}$ and so

$$H^2_{\text{ét,c,Hu}}(\mathbb{A}^1_C, \mathbf{Z}_p(1)) \simeq \big(\mathscr{O}_{\mathbb{P}^1,\infty}/C\big) \oplus \mathbf{Z}_p.$$

Hence the ℓ-adic compactly supported cohomology groups behave very differently in the cases $\ell = p$ and $\ell \neq p$, where $H^2_{\text{ét,c,Hu}}(\mathbb{A}^1_C, \mathbf{Z}_\ell(1)) \simeq \mathbf{Z}_\ell$. Note also that the action of $G(\mathbf{Q}_p)$ on $H^2_{\text{ét,c,Hu}}(\mathbb{A}^1_C, \mathbf{Z}_p(1))$ is not smooth, contrary to the case $\ell \neq p$.

A.2.2. *Continuous compactly supported étale cohomology.* We will also study a different version of Huber's compactly supported cohomology: For X as in Section A.2.1, we define its (continuous) compactly supported cohomology by:

$$\mathrm{R}\Gamma_{\mathrm{c}}(X_{\text{ét}}, (F_n)_n) := \mathrm{R}\varprojlim_n \mathrm{R}\Gamma_c(X_{\text{ét}}, F_n).$$

We have

$$\Gamma_{\mathrm{c}}(X_{\text{ét}}, (F_n)_n) = \{(s_n)_n \in \varprojlim_n \Gamma(X_{\text{ét}}, F_n) | \mathrm{supp}(s_n) \text{ is proper}\}.$$

The following properties are obtained directly from the definition and the corresponding properties for the compactly supported cohomology of F_n's.

(1) There is an exact sequence

$$0 \to \mathrm{R}^1\varprojlim_n H^{i-1}_{\text{ét},c}(X, F_n) \to H^i_{\text{ét,c}}(X, (F_n)_n) \to \varprojlim_n H^i_{\text{ét},c}(X, F_n) \to 0$$

(2) Let U be a taut open subspace of X, let $Z = X \setminus U$, and let $i : Z \hookrightarrow X$ be the inclusion. Then we have a distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{c}}(U_{\text{ét}}, (F_n|U)_n) \to \mathrm{R}\Gamma_{\mathrm{c}}(X_{\text{ét}}, (F_n)_n) \to \mathrm{R}\Gamma_{\mathrm{c}}(Z_{\text{ét}}, i^*(F_n)_n) \tag{A.4}$$

To lighten the notation, for $i \geq 0$, we will set

$$H^i_{\text{ét,c}}(X, \mathbf{Z}_p) := H^i_{\text{ét,c}}(X, (\mathbf{Z}/p^n)_n), \quad H^i_{\text{ét,c}}(X, \mathbf{Q}_p) := H^i_{\text{ét,c}}(X, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

References

[1] N. Abe, G. Henniart, F. Herzig, M.-F. Vignéras, *A classification of irreducible admissible mod p representations of p-adic reductive groups.* J. Amer. Math. Soc. 30 (2017), no. 2, 495–559.

[2] N. Abe, G. Henniart, M.-F. Vignéras, *Modulo p representations of reductive p-adic groups: Functorial properties.* Trans. Amer. Math. Soc. 371 (2019), no. 12, 8297–8337.

[3] Y. André, *Period mappings and differential equations. From $\mathbb{C}$ to $\mathbb{C}_p$. Tôhoku-Hokkaidô lectures in arithmetic geometry. With appendices by F. Kato and N. Tsuzuki.* MSJ Memoirs, 12. Math. Soc. of Japan, Tokyo, 2003.

[4] B. Bhatt, M. Morrow, P. Scholze, *Integral p-adic Hodge theory*, Publ. Math. de l'IHÉS 128 (2018), 219–397.

[5] B. Bhatt, M. Morrow, P. Scholze, *Topological Hochschild homology and integral p-adic Hodge theory*, Publ. Math. de l'IHÉS 129 (2019), 199–310.

[6] V. Berkovich, *Vanishing cycles for formal schemes.* Invent. Math. 115 (1994), no. 3, 539–571.

[7] M. Borovoi, *Abelian Galois cohomology of reductive groups.* Mem. Amer. Math. Soc. 132 (1998), no. 626.

[8] F. Bruhat, J. Tits, *Groupes algébriques sur un corps local. Chapitre III. Compléments et applications à la cohomologie galoisienne.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 34 (1987), no. 3, 671–698.

[9] K. Česnavičius, T. Koshikawa, *The A_{inf}-cohomology in the semistable case*, Compositio Math. 155 (2019), no. 11, 2039-2128.

[10] M. Chen, L. Fargues, X. Shen, *On the structure of some p-adic period domains.* arXiv:1710.06935 [math.AG].

[11] P. Colmez, *Représentations de* $\mathrm{GL}_2(\mathbf{Q}_p)$ *et* (φ, Γ)*-modules.* Astérisque 330 (2010), 281–509.

[12] P. Colmez, G. Dospinescu, W. Nizioł, *Cohomology of p-adic Stein spaces.* Invent. Math. 219 (2020), 873–985.

[13] P. Colmez, G. Dospinescu, W. Nizioł, *Integral p-adic étale cohomology of Drinfeld symmetric spaces.* arXiv:1905.11495 [math.AG].

[14] P. Colmez, W. Nizioł, *On the cohomology of the affine space.* arXiv:1707.01133 [math.AG], to appear in p-adic Hodge theory (2017), Simons Symposia, Springer-Verlag.

[15] C. Curtis, G. Lehrer, J. Tits, *Spherical buildings and the character of the Steinberg representation.* Invent. Math. 58 (1980), 201–210.

[16] J. F. Dat, *Espaces symétriques de Drinfeld et correspondance de Langlands locale.* Ann. Scient. Éc. Norm. Sup, 39 (1), 1-74 (2006).

[17] F. Dat, S. Orlik, M. Rapoport, *Period domains over finite and local fields.* Cambridge Tracts in Mathematics, 183. Cambridge University Press, 2010.

[18] Y. Ding, *Simple L-invariants for* GL_n, Transactions of the American Mathematical Society, Vol. 372, No. 11, 7993-8042 (2019).

[19] M. Emerton, *Ordinary parts of admissible representations of p-adic reductive groups I. Definition and first properties.* Astérisque 331 (2010), 355–402.

[20] M. Emerton, *Ordinary parts of admissible representations of p-adic reductive groups II. Derived functors.* Astérisque 331 (2010), 403–459.

[21] G. Faltings, *Mumford-Stabilität in der algebraischen Geometrie.* Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 648–655, Birkhäuser, 1995.

[22] L. Fargues, *Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales.* Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales. Astérisque 291 (2004), 1–199.

[23] L. Fargues, *G-torseurs en théorie de Hodge p-adique.* Preprint.

[24] L. Fargues, J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique.* Astérisque 406 (2018) 51–382.

[25] J.-M. Fontaine, M. Rapoport, *Existence de filtrations admissibles sur des isocristaux.* Bull. Soc. Math. France 133 (2005), no. 1, 73–86.

[26] L. Gehrmann, *Automorphic L-invariants for reductive groups*, arXiv:1912.05209 [math.NT].

[27] U. Goertz, X. He, S. Nie, *Fully Hodge-Newton decomposable Shimura varieties.* arXiv:1610.05381v2 [math.AG], to appear in Peking Mathematical Journal.

[28] E. Grosse-Klönne, *On special representations of p-adic reductive groups.* Duke Math. J. 163 (2014), no. 12, 2179–2216.

[29] J. Hauseux, *Extensions entre séries principales p-adiques et modulo p de* $G(F)$. J. Inst. Math. Jussieu 15 (2016), no. 2, 225–270.

[30] J. Hauseux, *Compléments sur les extensions entre séries principales p-adiques et modulo p de* $G(F)$. Bull. Soc. Math. France 145 (2017), no. 1, 161–192.

[31] J. Hauseux, *Parabolic induction and extensions.* Algebra Number Theory 12 (2018), no. 4, 779–831.

[32] J. Hauseux, *On the exactness of ordinary parts over a local field of characteristic p.* Pacific J. Math. 295 (2018), no. 1, 17–30.

[33] J. Hauseux, *Sur une conjecture de Breuil–Herzig.* J. Reine Angew. Math. 751 (2019), 91–119.

[34] J. Hauseux, T. Schmidt, C. Sorensen, *Functorial properties of generalised Steinberg representations.* J. Number Theory 195 (2019), 312–329.

[35] F. Herzig, *The classification of irreducible admissible mod p representations of a p-adic* GL_n. Invent. Math. 186 (2011), no. 2, 373–434.

[36] R. Huber, *Étale cohomology of rigid analytic varieties and adic spaces.* Aspects of Mathematics, E30. Friedr. Vieweg & Sohn, 1996.

[37] R. Huber, *A comparison theorem for* ℓ*-adic cohomology.* Compositio Math. 112 (1998), no. 2, 217–235.

[38] K. Kedlaya, R. Liu, *Relative p-adic Hodge theory: Foundations.* Astérisque 371 (2015).

[39] G. R. Kempf, *Instability in Invariant Theory.* Annals of Math. 108 (1978), 299–316.

[40] M. Kneser, *Galois-Kohomologie halbeinfacher algebraischer Gruppen über p-adischen Körpern. I*, Math. Z. 88 (1965), 40–47.

[41] M. Kneser, *Galois-Kohomologie halbeinfacher algebraischer Gruppen über p-adischen Körpern. II.* Math. Z. 89 (1965), 250–272.

[42] R. E. Kottwitz, *Isocrystals with additional structure.* Compositio Math. 56 (1985), 201–220.

[43] R. E. Kottwitz, *Isocrystals with additional structure. II.* Compositio Math. 109 (1997), 255–339.

[44] J.-P. Labesse, *Cohomologie, stabilisation et changement de base.* Appendix A by L. Clozel, J.-P. Labesse, and Appendix B by L. Breen. Astérisque 257 (1999).

[45] K.-W. Lan, R. Liu, X. Zhu, *De Rham comparison and Poincaré duality for rigid varieties.* arXiv:1912.13030 [math.AG].
[46] T. Ly, *Représentations de Steinberg modulo p pour un groupe réductif sur un corps local.* Pacific J. Math. 277 (2015), no. 2, 425–462.
[47] J. S. Milne, *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field.* Cambridge Studies in Advanced Mathematics, vol. 170, Cambridge University Press, 2017.
[48] D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory.* Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2), 34. Springer-Verlag, 1994.
[49] S. Orlik, *Kohomologie von Periodenbereichen über endlichen Körpern.* J. Reine Angew. Math. 528 (2000), 201–233.
[50] S. Orlik, *The cohomology of period domains for reductive groups over finite fields.* Ann. Sci. École Norm. Sup. (4) 34 (2001), no. 1, 63–77.
[51] S. Orlik, *The cohomology of period domains for reductive groups over local fields.* Invent. Math. 162 (2005), no. 3, 523–549.
[52] S. Orlik, *On extensions of generalized Steinberg representations.* J. Algebra 293 (2005), no. 2, 611-630.
[53] S. Orlik, *The continuous cohomology of period domains over local fields.* Comment. Math. Helv. 82 (2007), no. 3, 665–681.
[54] S. Orlik, *The pro-etale cohomology of Drinfeld's upper half space.* arXiv:1908.10591 [math.NT].
[55] S. Orlik, M. Strauch, *On some properties of the functors $\mathscr{F}_P^G$ from Lie algebra to locally analytic representations*, arxiv.org/abs/1802.07514
[56] S. Ramanan, A. Ramanathan, *Some remarks on the instability flag.* Tohoku Math. J. (2) 36 (1984), no. 2, 269–291.
[57] M. Rapoport, *Period domains over finite and local fields.* Algebraic geometry – Santa Cruz 1995, 361–381, Proc. Sympos. Pure Math., 62, Part 1, Amer. Math. Soc., 1997.
[58] M. Rapoport, M. Richartz, *On the classification and specialization of F-isocrystals with additional structure.* Compositio Math. 103 (1996), no. 2, 153–181.
[59] M. Rapoport, E. Viehmann, *Towards a theory of local Shimura varieties.* Münster J. Math. 7 (2014), no. 1, 273–326.
[60] M. Rapoport, T. Zink, *Period spaces for p-divisible groups.* Annals of Mathematics Studies, 141, Princeton University Press, 1996.
[61] C. Riehm, *The norm* 1 *group of a p-adic division algebra.* Amer. J. Math. 92 (1970), 499–523.
[62] P. Schneider, U. Stuhler, *The cohomology of p-adic symmetric spaces.* Invent. Math. 105 (1991), 47–122.
[63] P. Scholze, J. Weinstein, *Berkeley lectures on p-adic geometry*, to appear in Annals of Mathematics Studies.
[64] R. Steinberg, *Regular elements of semisimple algebraic groups.* Inst. Hautes Études Sci. Publ. Math. No. 25 1965 49–80.
[65] J. Tits, *Classification of algebraic semisimple groups*, in *Algebraic Groups and Discontinuous Subgroups*, Proceedings of Symposia in Pure Mathematics, vol. 9, Amer. Math. Soc., 1966, 33–62.
[66] B. Totaro, *Tensor products in p-adic Hodge theory.* Duke Math. J. 83 (1996), no. 1, 79–104.
[67] M.-F. Vignéras, *The Right Adjoint of the Parabolic Induction*, in *Arbeitstagung Bonn 2013*, Progress in Mathematics, vol. 319, Birkhäuser, 2016, 405–425.

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