# Lecture 8: Reductive groups 1: absolute theory

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# Goal

(I) We summarise the key properties of algebraic groups needed for the study of automorphic forms. Full proofs of these results take many pages, thus we will focus on the statements (see the canonical books of Borel, Humphreys, Springer and the excellent notes of Conrad for proofs). In this lecture we only deal with the absolute theory, i.e. over an algebraically closed field.

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(II) We will use varieties "à la grand papa". I will assume some familiarity with basic algebraic geometry, but we will review the relevant points. Let $K$ be an **algebraically closed field of characteristic** 0 (many statements to come will fail or be more difficult to prove in positive characteristic!), e.g. $K = \mathbb{C}$.

# Affine varieties

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(II) Affine varieties over $K$ form a category, morphisms $f : X \to Y$ (with $X \subset K^n, Y \subset K^m$) being maps of the form $f(x) = (P_1(x), ..., P_m(x))$ with $P_i$ polynomial functions on $X$.

# Affine varieties

(I) Sending $X$ to the $K$-algebra $K[X] = \mathrm{Hom}_{var}(X, K)$ of **polynomial (or regular) functions on** $X$ gives an anti-equivalence between the category of affine varieties over $K$ and that of reduced finitely generated $K$-algebras, and evaluation at points of $X$ gives a bijection

$$X \simeq \mathrm{Hom}_{K-\mathrm{alg}}(K[X], K).$$

If $I_X \subset K[T_1, ..., T_n]$ is the ideal of polynomials vanishing on $X$, then (for $X \subset K^n$)

$$K[X] \simeq K[T_1, ..., T_n]/I_X.$$

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(II) If $S \subset X$ is a subset, its **Zariski closure** $\bar{S}$ is the smallest Zariski closed subset of $X$ containing $S$. A point $x \in X$ is in $\bar{S}$ if and only if $f(x) = 0$ for any $f \in K[X]$ vanishing on $S$.

# Affine varieties

(I) The variety $X$ is called connected if $X$ is not the disjoint union of two proper Zariski closed subsets. Equivalently, $K[X]$ has no nontrivial idempotent. The next result is highly nontrivial.

**Theorem** Let $X \subset \mathbb{C}^n$ be an affine variety. $X$ is connected for the Zariski topology if and only if $X$ is connected for the classical topology.

# Algebraic groups

(I) We can see (via $g \to (1/\det(g), g)$) $\mathbb{GL}_n(K)$ as the affine sub-variety

$$\mathbb{GL}_n(K) \simeq \{(t, X) \in K \times M_n(K) | \, t \det(X) = 1\}$$

of $K^{n^2+1} \simeq K \times M_n(K)$. An **algebraic subgroup of** $\mathbb{GL}_n(K)$ is a subgroup of $\mathbb{GL}_n(K)$ which is Zariski closed in $K^{n^2+1}$, i.e. a subgroup of $\mathbb{GL}_n(K)$ defined by polynomial equations in the coefficients and the inverse of the determinant of the matrices.

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(II) The term **algebraic group** refers to an algebraic subgroup of some $\mathbb{GL}_n(K)$. More conceptually, these are the group objects in the category of affine varieties over $K$.

(III) **Convention**: by **subgroup** of an algebraic group $G$ we wean a Zariski closed subgroup of $G$.

# Algebraic groups

(I) Here are a few very basic, but already not at all trivial results:

**Theorem (Borel, Chevalley, Zariski)**
a) If $f : G \to H$ is a morphism of algebraic groups, then $f(G)$ is Zariski closed in $H$ (hence an algebraic group!) and $f$ is an isomorphism if it is bijective.

b) If $G$ is an algebraic group, then the derived group $G_{\mathrm{der}}$ of $G$ is Zariski closed in $G$, thus an algebraic group. If $H$ is a Zariski closed **normal** subgroup of $G$, then $G/H$ is an algebraic group.

If $G$ is an algebraic group, its **neutral component** $G^0$ is the connected component containing $1 \in G$. It is a normal closed subgroup of $G$, of finite index.

# Examples of algebraic groups: tori

(I) A key example of algebraic group is the group $\mathbb{G}_m^n$ of diagonal matrices in $\mathbb{GL}_n(K)$. An algebraic group isomorphic to $\mathbb{G}_m^n$ is called a **torus of rank** $n$. The **character group** functor

$$X : \{K-\text{tori}\} \to \{\text{finite free } \mathbb{Z}-\text{modules}\},$$

$$X(T) = \mathrm{Hom}_{\mathrm{alg.gr}}(T, \mathbb{G}_m)$$

induces an anti-equivalence between $K$-tori and finite free $\mathbb{Z}$-modules. A quasi-inverse associates to a finite free $\mathbb{Z}$-module $M$ the torus $T = \mathrm{Hom}_{\mathrm{gr}}(M, K^*)$.

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(II) One checks that $\mathrm{End}_{\mathrm{alg.gr}}(\mathbb{G}_m) = \mathbb{Z}$ (where $n \in \mathbb{Z}$ corresponds to the character $t \to t^n$ of $K^* = \mathbb{G}_m$). Letting $X_*(T) := \mathrm{Hom}_{\mathrm{alg.gr}}(\mathbb{G}_m, T)$ be the group of **cocharacters** of $T$, there is a canonical perfect duality,

$$\langle\ .\ \rangle : X(T) \times X_*(T) \to \mathrm{End}_{\mathrm{alg.gr}}(\mathbb{G}_m) = \mathbb{Z},\ \langle u, v\rangle = u \circ v.$$

# More examples of algebraic groups

(I) Other standard examples of algebraic groups include $\mathbb{SL}_n(K)$, orthogonal groups associated to quadratic spaces over $K$, symplectic groups (attached to non-degenerate alternating forms) etc.

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(II) Somewhat more exotic but very important: if $D$ is a finite dimensional division algebra over $K$, then $D^*$ (the group of units of $D$) is an algebraic group. This will appear a lot in the second semester, when $D$ is a quaternion algebra. One can also consider the units with reduced norm 1 and get another algebraic group.

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(III) Finite groups are algebraic!

## Unipotent/solvable groups

(I) Another key example is the subgroup $U_n(K)$ of upper triangular unipotent matrices in $\mathbb{GL}_n(K)$. An algebraic group isomorphic to a subgroup of $U_n(K)$ is called an **unipotent group**. If $G \subset \mathbb{GL}_n(K)$ is an algebraic subgroup, then $G$ is unipotent if and only if all matrices in $G$ are unipotent, or equivalently $G$ can be conjugated to land in $U_n(K)$.

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(II) Yet another fundamental example is the subgroup $B_n(K)$ of upper triangular matrices in $\mathbb{GL}_n(K)$. It is the semi-direct product of the diagonal torus $D_n(K) \simeq \mathbb{G}_m^n$ and of $U_n(K)$ and it is a solvable connected group.

Theorem (Lie-Kolchin) Any connected solvable subgroup $G \subset \mathbb{GL}_n(K)$ has a conjugate contained in $B_n(K)$.

# Borel subgroups

(I) A maximal connected solvable (algebraic) subgroup $B$ of an algebraic group $G$ is called a **Borel subgroup**. Their study is at the basis of almost all key results about algebraic groups. Some of the main results of the theory are summarised in the following hard theorem:

Theorem (Borel) Let $G$ be a connected algebraic group.
a) All maximal tori and all Borel subgroups in $G$ are $G$-conjugate.

b) $G$ is the union of all of its Borel subgroups. The union of all maximal tori is the set of semisimple elements of $G$, i.e. those $g \in G$ which are diagonalisable matrices under some (equivalently any) embedding $G \subset \mathbb{GL}_n(K)$.

c) $G$ is unipotent if and only if $G$ has no nontrivial torus, and $G$ is a torus if and only if any $g \in G$ is semisimple.

# Algebraic groups and Lie algebras

(I) Any algebraic group $G \subset \mathbb{GL}_n(K)$ is a smooth variety and the tangent space to $G$ at 1 has a natural structure of $K$-Lie algebra. This is the Lie algebra $\mathfrak{g}$ of $G$. If $I \subset K[\mathbb{GL}_n(K)]$ is the ideal of $G$, then

$$\mathfrak{g} = \{A \in M_n(K) | \frac{df}{dt}|_{t=0} f(1 + tA) = 0, \ \forall f \in I\},$$

with Lie bracket $[X, Y] = XY - YX$.

# Algebraic groups and Lie algebras

(I) In practice one uses the following description to actually compute $\mathfrak{g}$:

$$\mathfrak{g} = \ker(G(K[\varepsilon]) \to G(K)),$$

where $K[\varepsilon] = K[T]/T^2$ is the ring of dual numbers and $G(K[\varepsilon]) \to G(K)$ is induced by the map sending $T$ to 0.

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(II) If $K = \mathbb{C}$ and $G \subset \mathbb{GL}_n(\mathbb{C})$ is an embedding, then $G(K)$ is naturally a Lie group and $\mathfrak{g}$ is the Lie algebra of that group:

$$\mathfrak{g} = \{X \in M_n(\mathbb{C}) | \, e^{tX} \in G \ , \forall t \in \mathbb{R}\}.$$

## Representations of algebraic groups

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(II) A very important example is the **adjoint representation** (say $G \subset \mathbb{GL}_n(K)$)

$$\mathrm{Ad} : G \to \mathbb{GL}(\mathfrak{g}), \ \mathrm{Ad}(g)(X) = gXg^{-1}.$$

Its tangent map at 1 is

$$ad = d(\mathrm{Ad}) : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \ \mathrm{ad}(X)(Y) = [X, Y] := XY - YX.$$

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(III) If $V$ is a finite dimensional representation of a torus $T$, then $V = \oplus_{a \in X(T)} V_a$, where $V_a = \{v \in V | \, t.v = a(t)v,\ \forall t \in T\}$ is the $a$-**weight space**. We say that $a \in X(T)$ is a **weight of** $V$ if $V_a \neq 0$.

## Representations of algebraic groups

(I) More generally, a $K$-linear action of $G$ on a $K$-vector space $V$ is called algebraic (or simply a **representation of** $G$) if $V$ is a union of finite dimensional $G$-representations.

# Representations of algebraic groups

(I) More generally, a $K$-linear action of $G$ on a $K$-vector space $V$ is called algebraic (or simply a **representation of** $G$) if $V$ is a union of finite dimensional $G$-representations.

(II) Such infinite dimensional representations of $G$ arise from actions of $G$ on varieties $X$, i.e. abstract actions for which the natural map $G \times X \to X$ is a morphism of varieties. Then $G$ acts on $K[X]$ by $g.f(x) = f(g^{-1}x)$.

Theorem If $G$ acts on a variety $X$, then $K[X]$ is a representation of $G$ and the action of $G$ on $X$ can be linearized: there is a closed embedding $X \subset K^n$ (for some $n$) and a representation $\rho : G \to \mathbb{GL}_n(K)$ such that $g.x = \rho(g)(x)$ for $g \in G, x \in X$.

In particular $K[G]$ is a representation of $G$, via $g.f(g') = f(g^{-1}g')$ (or via $g.f(g') = f(g'g)$...).

# Reductive groups

(I) We write $\mathrm{Rep}^{\mathrm{alg}}(G)$ for the category of representations of $G$ and $\mathrm{Irr}(G)$ for set of isomorphism classes of irreducible objects of $\mathrm{Rep}^{\mathrm{alg}}(G)$. All $V \in \mathrm{Irr}(G)$ are finite dimensional (by definition!).

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(II) We say that $G$ is **reductive** if every $V \in \mathrm{Rep}^{\mathrm{alg}}(G)$ is a direct sum of irreducible representations, or equivalently if any $G$-stable subspace of $V$ has a $G$-stable complement. Yet another equivalent condition is that the natural evaluation map is an isomorphism

$$\bigoplus_{\pi \in \mathrm{Irr}(G)} \pi \otimes_K \mathrm{Hom}_G(\pi, V) \simeq V.$$

# Reductive groups

(I) It follows easily from the last point that when $G$ is reductive the functor $V \to V^G$ is exact on $\mathrm{Rep}^{\mathrm{alg}}(G)$ and

$$K[G] \simeq \bigoplus_{\pi \in \mathrm{Irr}(G)} \pi \otimes_K \pi^*.$$

Conversely, if $K[G]$ is semi-simple, or equivalently if the above isomorphism holds, then $G$ is reductive.

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Conversely, if $K[G]$ is semi-simple, or equivalently if the above isomorphism holds, then $G$ is reductive.

(II) This should ring a bell: it looks very similar to the case of finite groups, more generally of compact groups! We will see that this is not a coincidence...

# Reductive groups

(I) Here is an extremely beautiful characterisation of reductive groups:

**Theorem (Hilbert, Popov)** An algebraic group $G$ is reductive if and only if $K[X]^G$ is a finitely generated $K$-algebra for any action of $G$ on an affine variety $X$.

This allows one to construct quotients for actions of reductive groups $G$ on affine varieties $X$: since $K[X]^G$ is finitely generated and reduced, it is the algebra of regular functions on some affine variety $X//G$ (called the categorical or GIT quotient of $X$ by $G$). The inclusion $K[X]^G \to K[X]$ gives rise to a $G$-invariant surjective morphism $\pi : X \to X//G$, identifying $X//G$ with the set of closed orbits of elements of $X$.

# Reductive groups

(I) More precisely, if $Y, Z$ are closed disjoint sub-varieties of $X$ stable under $G$, then we can find $f \in K[X]^G$ vanishing on $Y$ and equal to 1 on $Z$. It follows that each fibre of $\pi_X$ contains a unique closed $G$-orbit.

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(II) If $H$ is a closed and reductive subgroup of an algebraic group, then $H$ acts on $G$ by $h.g = gh^{-1}$,and the fibres of $\pi_G : G \to G//H$ are the $H$-orbits in $G$, thus $\pi$ gives a bijection $G/H \simeq G//H$, endowing $G/H$ with a structure of affine variety.

## Examples of reductive groups

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(II) It is not hard to show that $G$ is reductive if and only if $G^0$ is reductive, that the product of reductive groups is reductive.

(III) Reductivity is preserved under passage to **normal** (Zariski closed) subgroups and to quotients by such subgroups, but not stable under passage to Zariski closed subgroups (otherwise any algebraic group would be reductive...).

## The "correct" definition of reductive groups

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(II) Any algebraic group has a largest unipotent normal subgroup, called its **unipotent radical**. One implication in the following deep theorem follows from the previous remark, the other one is much harder and uses many key structural results about reductive groups:

Theorem An algebraic group $G$ is reductive if and only if its unipotent radical is trivial.

In positive characteristic the right definition of reductivity is the triviality of the unipotent radical. With that definition, everything that we will say about the structure and classification of reductive groups will work in positive characteristic.

# Dévissage of algebraic groups

(I) The next deep theorem of Mostow (false in positive characteristic) reduces the study of general algebraic groups to unipotent and reductive groups:

Theorem (Mostow) Any algebraic group $G$ with unipotent radical $U$ is the semi-direct product of $U$ and of a reductive group $H$, which is unique up to conjugacy by $U$.

Such a group $H$ is called a **Levi subgroup** of $G$.

# Dévissage of reductive groups

(I) A reductive group $G$ is called **semisimple** if $G$ has finite center. A key example of semisimple group is $\mathbb{SL}_n(K)$. Others are $\mathbb{PGL}_n(K)$, symplectic groups, special orthogonal groups, etc. The next result is not at all trivial:

Theorem Let $G$ be a reductive group.
a) $G_{\mathrm{der}}$ is a semisimple group and $Z(G)^0$ is a torus.

b) If $G$ is connected, then $G_{\mathrm{der}}$ is perfect, i.e. equal to its own derived group, and $G = Z(G)^0 G_{\mathrm{der}}$, the intersection being finite (by a)).

In particular we have $X(G) = \{1\}$ for any connected semisimple group $G$, where for any algebraic group $G$ we let

$$X(G) = \mathrm{Hom}_{\mathrm{alg.gr}}(G, \mathbb{G}_m).$$

# Root data

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(II) From now on we fix a connected reductive group $G$ and a maximal torus $T \subset G$, i.e. a torus in $G$ not strictly contained in any other torus of $G$. Let $X = X(T)$ be the character group of $T$. Then $Z_G(T) = T$ (not easy!) and the adjoint action of $T$ on $\mathfrak{g} = \mathrm{Lie}(G)$ diagonalizes and gives a decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \oplus_{a \in \Phi} \mathfrak{g}_a,$$

for some finite (maybe empty) subset $\Phi = \Phi(G, T) \subset X \setminus \{0\}$. So $\Phi$ consists of the nontrivial weights of this representation of $T$ on $\mathfrak{g}$.

# Root data

(I) The corresponding weight space

$$\mathfrak{g}_a = \{Z \in \mathfrak{g} | \operatorname{Ad}(t)(Z) = a(t)Z, \ \forall t \in T\}$$

is called the $a$-root space and the elements of $\Phi$ are called **roots** of $(G, T)$. The name comes from the equality

$$\det(u_{\bullet}\mathrm{id} - \operatorname{Ad}(t)) = (u-1)^{\dim T} \prod_{a \in \Phi} (X - a(t))^{\dim \mathfrak{g}_a}.$$

# Root data

(I) One of the key nontrivial results of the theory is the following result (in which $-a$ is the character $t \to 1/a(t)$):

**Theorem** For any $a \in \Phi$ we have $\mathbb{Q}a \cap \Phi = \{\pm a\}$, $\dim \mathfrak{g}_a = 1$ and

• there is a unique closed subgroup $U_a \subset G$ (the $a$-**root group**) normalized by $T$ and such that $\mathrm{Lie}(U_a) = \mathfrak{g}_a$.

• there is a unique $a^\vee \in X_*(T)$ such that $\langle a, a^\vee \rangle = 2$ and such that $s_a(x) := x - \langle x, a^\vee \rangle a$ permutes $\Phi$.

The set $\Phi^\vee = \{a^\vee | \, a \in \Phi\} \subset X_*(T)$ is called the set of **coroots** of $(G, T)$.

# Root data

(I) Where does $a^\vee$ come from? A fairly hard theorem ensures the existence of a homomorphism of algebraic groups $\varphi_a : \mathbb{SL}_2(K) \to G$, uniquely determined up to $T$-conjugacy, such that $\varphi_a$ sends the diagonal torus in $\mathbb{SL}_2(K)$ into $T$ and induces isomorphisms

$$\varphi_a : \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \simeq U_a, \ \varphi_a : \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \simeq U_{-a}.$$

The following cocharacter $a^\vee$ of $T$ is independent of the choice of $\varphi_a$:

$$a^\vee(x) = \varphi_a \left( \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} \right).$$

Moreover, one checks that $\langle a, a^\vee \rangle = 2$.

# Root data

(I) OK, all this deserves some examples. Take $G = \mathbb{SL}_2(K)$ with the diagonal torus $T = \{x \to \lambda(x) = \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}\}$, then

$$\mathrm{Ad}(\lambda(x))e = x^2 e, \mathrm{Ad}(\lambda(x))f = x^{-2} f,$$

where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

thus the roots are $\pm a$, with $a(\begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}) = x^2$. The root spaces are $\mathfrak{g}_a = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \mathfrak{g}_{-a} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$, the root groups are $U_a = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, U_{-a} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$, and $a^\vee(x) = \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}$.

# Root data

(I) Now take $G = \mathbb{GL}_n(K)$ with the diagonal torus $T = \{\mathrm{diag(x_1, ..., x_n)} | \, \mathrm{x_i \in K^*}\}$. Then

$$X = X(T) = \oplus_{i=1}^{n} \mathbb{Z} e_i, \;\; e_i(\mathrm{diag}(x_1, ..., x_n)) := x_i$$

and ($x$ is in position $i$)

$$X_*(T) = \oplus_{i=1}^{n} \mathbb{Z} e_i^\vee, \;\; e_i^\vee(x) := \mathrm{diag}(1, ..., 1, x, 1, ..., 1).$$

The set of roots is then (exercise)

$$\Phi = \{e_i - e_j | \, 1 \leq i \neq j \leq n\}$$

the root spaces are $\mathfrak{g}_{e_i - e_j} = K \bullet E_{ij}$ and $(e_i - e_j)^\vee = e_i^\vee - e_j^\vee$.

# Root data

(I) Upshot: $(G, T)$ gives rise to a **root datum**, i.e. a 4-tuple $(X, \Phi, X^\vee, \Phi^\vee)$ together with a (necessarily uniquely determined) bijection $\Phi \to \Phi^\vee$, $a \to a^\vee$, where:

- $X, X^\vee$ are finite free $\mathbb{Z}$-modules, together with a perfect pairing $\langle\, ,\, \rangle : X \times X^\vee \to \mathbb{Z}$.

- $\Phi \subset X \setminus \{0\}$ and $\Phi^\vee \subset X^\vee \setminus \{0\}$ are finite subsets, stable under $a \to -a$, satisfying $\langle a, a^\vee \rangle = 2$ for $a \in \Phi$, and such that the map $s_a : X \to X$, $s_a(x) = x - \langle x, a^\vee \rangle a$ permutes $\Phi$ and $s_{a^\vee} : X^\vee \to X^\vee$, $s_{a^\vee}(\lambda) = \lambda - \langle a, \lambda \rangle a^\vee$ permutes $\Phi^\vee$. The root datum is called **reduced** if $\mathbb{Q}a \cap \Phi = \{\pm a\}$ for all $a \in \Phi$.

# Root data

(I) Here's now the mindblowing theorem (which actually holds in positive characteristic as well):

Theorem (Chevalley, Demazure) There is a canonical bijection

$$\{\text{reductive groups over } K\}/\simeq \; \to \{\text{reduced root data}\}/\simeq .$$

If we exchange $X$ and $X^\vee$, as well as $\Phi$ and $\Phi^\vee$, we get a new root datum, to which the theorem associates a reductive group (up to isomorphism) $G^\vee$, called the **Langlands dual group of** $G$.

# Root data

(I) Somewhat more precisely: any reduced root datum is isomorphic to the one of a pair $(G, T)$, and any isomorphism between the root data of $(G, T)$ and $(G', T')$ arises from an isomorphism $(G, T) \simeq (G', T')$, unique up to the conjugacy action of $T$ and $T'$. Thus the reduced datum $RD(G, T)$ of a pair $(G, T)$ determines the pair uniquely up to isomorphism.

# Root data

(I) Any root datum $(X, \Phi, X^\vee, \Phi^\vee)$ gives rise to an abstract **root system** $(V, \Phi)$, where $V = \mathbb{Q}\bullet\Phi \subset X_{\mathbb{Q}} := X \otimes_{\mathbb{Z}} \mathbb{Q}$.

# Root data

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(II) This simply means that $\Phi$ is a finite spanning subset of $V$, not containing 0 and such that for any $a \in \Phi$ there is a linear form $l \in V^*$ such that $l(a) = 2$, $l(\Phi) \subset \mathbb{Z}$ and the reflection $s_a(x) = x - l(x)a$ permutes $\Phi$. The root system is called **reduced** if $\mathbb{Q}a \cap \Phi = \{\pm a\}$ for $a \in \Phi$.

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(III) Any root system $(\Phi, V)$ has an associated **Weyl group** $W(\Phi)$, a finite subgroup of $\mathbb{GL}(V)$ generated by the $s_a$ for $a \in \Phi$.

## Root data

(I) If $\Phi = \Phi(G, T)$ for some pair $(G, T)$, then

$$Z(G) = \cap_{a \in \Phi} \ker(a)$$

and so $G$ is semisimple if and only if $\Phi$ spans $X_{\mathbb{Q}}$, in which case $(\Phi, X_{\mathbb{Q}})$ is a root system, with Weyl group

$$W(\Phi) \simeq N_G(T)/T.$$

We call this $W(G, T)$ or simply $W(G)$ (all $W(G, T)$ are isomorphic to each other), the **Weyl group of** $G$. For instance $W(\mathbb{GL}_n(K)) \simeq S_n$.

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(II) For a general connected reductive $G$ there is a bijection between maximal tori in $G$ and $G_{\mathrm{der}}$, inducing a bijection between sets of roots and an isomorphism of Weyl groups.

# Root data

(I) Since root data encode all the information about the group, they should also encode information about Borel subgroups. This goes as follows. For any Borel subgroup $B$ containing the fixed maximal torus $T$, $T$ acts on $\mathfrak{b} = \mathrm{Lie}(B)$, giving rise to a decomposition

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{a \in \Phi(B,T)} \mathfrak{g}_a$$

for some subset $\Phi(B, T)$.

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for some subset $\Phi(B, T)$.

(II) The subset $\Phi^+ = \Phi(B, T)$ of $\Phi$ has the property that $\Phi = \Phi^+ \coprod -\Phi^+$ and $\Phi^+$ is closed, in the sense that $a + b \in \Phi^+$ whenever $a, b, a + b \in \Phi$ and $a, b \in \Phi^+$. We call such subsets $\Phi^+$ **systems of positive roots**.

## Reductive groups in terms of combinatorial data

(I) For any reduced root system $(\Phi, V)$ the systems of positive roots are exactly the subsets of the form $\Phi^+ = \{a \in \Phi \mid l(a) > 0\}$ for some $l \in V^*$ not vanishing on any $a \in \Phi$ and $W(\Phi)$ permutes them simply transitively.

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(II) The key theorem for reductive groups:

**Theorem** For a connected reductive group $G$ with a maximal torus $T$ and $\Phi = \Phi(G, T)$, the map $B \to \Phi(B, T)$ gives a bijection between Borel subgroups of $G$ containing $T$ and systems of positive roots in $\Phi$. The Weyl group of $(G, T)$ permutes these sets simply transitively.

If $\Phi^+$ is a system of positive roots, the associated Borel subgroup is the subgroup generated by $T$ and by the $U_a$ for $a \in \Phi^+$.

# Reductive groups in terms of combinatorial data

(I) For instance, for $G = \mathbb{GL}_n(K)$ the upper triangular Borel subgroup corresponds to the system of positive roots $\{e_i - e_j | \, i < j\}$ (note that the root system of $G$ lives in the vector space $\{v = (v_1, ..., v_n) \in \mathbb{Q}^n | \, \sum v_i = 0\}$).

# Root systems

(I) Any reduced system of positive roots $\Phi^+$ has a unique **base (or system of simple roots)** $\Delta$, i.e. a subset of $\Phi$ which is a basis of $V$ and such that any root $a$ can be written $a = \sum_{b \in \Delta} n_b b$ with $n_b$ integers of the same sign. We recover $\Phi^+$ from $\Delta$ as those linear combinations in which all $n_b \geq 0$.

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(II) Actually $\Delta$ consists of those roots in $\Phi^+$ which cannot be expressed as the sum of two roots in $\Phi^+$, $\Phi = \cup_{w \in \Delta} W.\Delta$ (where $W$ is the Weyl group) and $W$ is generated by $s_a$ with $a \in W$, and $G$ is generated by $T$ and the $U_a$ for $a \in \Delta$.

## Compactness and reductivity

(I) We start by observing the following easy, but crucial result:

**Theorem** The Zariski closure of any compact subgroup of $\mathbb{GL}_n(\mathbb{C})$ is reductive.

Let $K \subset \mathbb{GL}_n(\mathbb{C})$ and $G$ the Zariski closure of $K$. If $V$ is a finite dimensional algebraic representation of $G$, by the unitary trick there is a hermitian inner product on $V$ invariant under $K$. Take a subspace $W$ of $V$ stable under $G$, then $W^\perp$ is stable under $K$. Since the representation is algebraic and $K$ is Zariski dense, $W^\perp$ is stable under $G$ and $V$ is semisimple, thus $G$ is reductive!

# Compactness and reductivity

(I) Using the above theorem and the polar decomposition, we are able to prove:

**Theorem** $\mathbb{GL}_n(\mathbb{C})$ is reductive.

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Theorem $\mathbb{GL}_n(\mathbb{C})$ is reductive.

(II) Consider the **Cartan involution**

$$\theta : \mathbb{GL}_n(\mathbb{C}) \to \mathbb{GL}_n(\mathbb{C}), x \to (x^*)^{-1},$$

where $x^*$ is the complex conjugate of the transpose of $x$. The induced map on Lie algebras is still denoted $\theta : M_n(\mathbb{C}) \to M_n(\mathbb{C}), X \to -X^*$. Note that

$$\mathbb{GL}_n(\mathbb{C})^{\theta=1} = U(n) = \{g \in \mathbb{GL}_n(\mathbb{C}) |\, gg^* = 1\}$$

is compact and $M_n(\mathbb{C})^{\theta=-1}$ is the space of hermitian matrices.

## Compactness and reductivity

(I) The classical polar decomposition asserts that the map

$$U(n) \times M_n(\mathbb{C})^{\theta=-1} \to \mathbb{GL}_n(\mathbb{C}),\ (k, X) \to k \exp(X)$$

is a homeomorphism (even diffeomorphism).

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(II) We claim that $U(n)$ is Zariski dense in $\mathbb{GL}_n(\mathbb{C})$, i.e. a polynomial function $f \in \mathbb{C}[\mathbb{GL}_n(\mathbb{C})]$ vanishing on $U$ vanishes everywhere. By the polar decomposition we need to show that $f(k \exp(X)) = 0$ for $k \in U(n)$ and $X$ hermitian. But the map $z \to f(k \exp(zX))$ is holomorphic and vanishes on $i\mathbb{R}$ since $\exp(i\mathbb{R}X) \subset U(n)$, thus it is the zero map and so $f(k \exp(X)) = 0$.

# Compactness and reductivity

(I) An algebraic subgroup $G \subset \mathbb{GL}_n(\mathbb{C})$ is called **self-adjoint** if $G$ is stable under $g \to g^*$, i.e. under $\theta$.

Theorem If an algebraic group $G \subset \mathbb{GL}_n(\mathbb{C})$ is the Zariski closure of some compact subgroup $K \subset \mathbb{GL}_n(\mathbb{C})$, then $G$ is conjugated to a self-adjoint group.

By conjugating $G$, we may assume that $K \subset U(n)$, and we will prove that $G$ is self-adjoint. Pick $g \in G$ and $f \in \mathbb{C}[\mathbb{GL}_n(\mathbb{C})]$, we want to show that $f(g^*) = 0$. But $f$ vanishes on $G$, thus on $K$, and $f(g^*) = f(g^{-1}) = 0$ for $g \in K$. By Zariski density of $K$ we obtain $f(g^*) = 0$ for $g \in G$.

# Compactness and reductivity

(I) We will prove now the converse of the previous theorem. So let $G$ be self-adjoint and let $\theta : G \to G$ be the restriction of the Cartan involution, and $\theta : \mathfrak{g} = \mathrm{Lie}(G) \to \mathfrak{g}$ its derivative. Letting

$$\mathfrak{p} = \mathfrak{g}^{\theta=-1}, \ \mathfrak{k} = \mathfrak{g}^{\theta=1},$$

we have

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we have

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}.$$

(II) Let $K = G^{\theta=1}$, a compact subgroup of $G$ (closed in $U(n)$) with $\mathrm{Lie}(K) = \mathfrak{g}^{\theta=1} = \mathfrak{k}$ and

$$\mathfrak{p} = i\mathfrak{k} = i\mathrm{Lie}(K).$$

## Compactness and reductivity

(I) The fundamental result is then:

Theorem (Cartan, Chevalley) For $G$ as above the map

$$K \times \mathfrak{p} \to G, \ (k, X) \to k \exp(X)$$

is a homeomorphism (even a diffeomorphism) and $K$ is a maximal compact subgroup of $G$, Zariski dense in $G$.

The decomposition $G = K \exp(\mathfrak{p})$ is called the **Cartan decomposition**. It follows from it that $G = KG^0$, i.e. $K$ meets each connected component of $G$.
The tricky point in the proof is proving that if $g = k \exp(X) \in G$, with $k \in U(n)$ and $X$ hermitian, then $k \in K$ and $X \in \mathfrak{p}$. It suffices to check that $X \in \mathfrak{p}$, and actually that $e^{tX} \in G$ for all $t \in \mathbb{R}$.

# The Cartan-Chevalley-Mostow theorem

(I) Since $G$ is stable under $\theta$, we have $\theta(g) = ke^{-X} \in G$, thus $e^{2X} \in G$ and so $e^{2nX} \in G$ for all $n \in \mathbb{Z}$. Pick a polynomial $P$ vanishing on $G$, then $P(e^{2nX}) = 0$ for all $n \in \mathbb{Z}$, and an easy (but great!) exercise shows that $P(e^{tX}) = 0$ for all $t \in \mathbb{R}$. Varying $P$ yields $e^{tX} \in G$ for all $t \in \mathbb{R}$.

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(II) If $L$ is a compact subgroup of $G$ containing $K$ strictly, by the Cartan decomposition there is $X \in \mathfrak{p}$ nonzero with $e^X \in L$, but then $e^{kX}$ stay in the compact set $L$ for $k \in \mathbb{Z}$, impossible (diagonalize $X$!). Thus $K$ is a maximal compact subgroup. The argument that $K$ is Zariski dense in $G$ is identical to the one for $\mathbb{GL}_n(\mathbb{C})$.

# What is an algebraic group?

(I) Here comes the amazing theorem:

Theorem (Cartan, Chevalley, Mostow) For any reductive group $G \subset \mathbb{GL}_n(\mathbb{C})$ there is $g \in \mathbb{GL}_n(\mathbb{C})$ such that $gGg^{-1}$ is self-adjoint.

Combining everything:

Theorem (Cartan, Chevalley, Mostow) a) The reductive groups over $\mathbb{C}$ are precisely the Zariski closures in $\mathbb{GL}_n(\mathbb{C})$ (for some $n$) of compact subgroups of $\mathbb{GL}_n(\mathbb{C})$.

b) Any reductive group $G$ has a unique conjugacy class of maximal compact subgroups $K$, and $K$ is Zariski dense in $G$.

# What is an algebraic group?

(I) Moreover, restriction to $K$ induces an equivalence of categories between finite dimensional objects of $\mathrm{Rep}^{\mathrm{alg}}(G)$ and of $\mathrm{Rep}(K)$ and identifies

$$\mathbb{C}[G] \simeq L^2(K)^{K-\mathrm{fin}}.$$