Lecture 7: L-functions and Hecke operators

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Goal

(I) In a first part we will prove that many interesting L-functions attached to modular forms have analytic continuation and functional equations, by pushing the argument for the meromorphic continuation of real analytic Eisenstein series (for $\mathbb{SL}_2(\mathbb{Z})$) to its limits.

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(II) In a second part we introduce a class of operators on spaces of modular forms, which play a fundamental role and allow one to isolate a class of very nice modular forms, whose L-functions behave like those of Dirichlet characters, for instance they have interesting Euler product factorisations. These modular forms have amazing arithmetic properties.

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(I) We will work at level $\Gamma = \Gamma_0(N)$, for some positive integer N, where

$$\Gamma_0(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{SL}_2(\mathbb{Z}) |\, \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N}\}.$$

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$$[\Gamma(1) : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1}).$$

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$$[\Gamma(1) : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1}).$$

(III) We let $M_k(N) = M_k(\Gamma_0(N))$ and $S_k(N) = S_k(\Gamma_0(N))$. If $N = 1$ we simply write M_k, S_k.

Mellin transform

(I) The general machinery for analytic continuations and functional equations is quite simple. Let $\Phi \in C^\infty((0,\infty))$ and $c, v > 0$ be such that

$$\Phi(t) = O(e^{-ct}), t \to \infty, \; \Phi(t) = O(t^{-v-1}), \; t \to 0.$$

The **Mellin transform** of Φ is defined by

$$M\Phi(s) = \int_0^\infty \Phi(t) t^s \frac{dt}{t}.$$

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It is holomorphic in $\mathrm{Re}(s) > 1 + v$.

(II) In practice we take $\Phi(t) = \sum_{n\geq 1} a_n e^{-nct}$ with $c > 0$, $a_n = O(n^v)$. In this case we have the crucial identity

$$M\Phi(s) = c^{-s}\Gamma(s)L(s),\ \mathrm{Re}(s) > 1 + v,$$

$$L(s) := \sum_{n\geq 1} \frac{a_n}{n^s},\ \Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x} = M(-\exp)(s).$$

Mellin transform

(I) In practice the interesting function is $s \to L(s)$, and the previous identity (plus classical properties of Γ, for instance $1/\Gamma$ extends to a holomorphic function on $\mathbb{C}$, vanishing at negative integers) allows one to study $L(s)$ via $M\Phi(s)$.

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(II) The Mellin transform is a version of Fourier transform, thus one expects to recover Φ from $M\Phi$. This can be done, thanks to:

Theorem For all $c > 1 + v$ we have

$$\Phi(t) = \frac{1}{2i\pi} \int_{\mathrm{Re}(s)=c} M\Phi(s) t^{-s} ds.$$

Mellin transform

(I) The proof is simple: letting $f(x) = e^{cx}\Phi(e^x)$, we have

$$M\Phi(c+iu) = \int_0^\infty \Phi(t) t^{c+iu} \frac{dt}{t} =$$

$$\int_{\mathbb{R}} \Phi(e^x) e^{x(c+iu)} dx = \int_{\mathbb{R}} f(x) e^{ixu} dx.$$

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(II) For many Φ one can extend $M\Phi$ meromorphically to $\mathbb{C}$ using the same trick as the one used for Eisenstein series in the previous lecture:

$$M\Phi(s) = \int_0^1 \varphi(t)t^s\frac{dt}{t} + \int_1^\infty \varphi(t)t^s\frac{dt}{t} =$$

$$= \int_1^\infty (\varphi(t^{-1})t^{-s} + \varphi(t)t^s)\frac{dt}{t}.$$

Mellin transform

(I) If $\varphi(t^{-1})$ behaves well enough the integral converges absolutely for all $s \in \mathbb{C}$ and the resulting function is holomorphic in s. If there are $k \in \mathbb{R}, c \in \{-1, 1\}$ such that $\varphi(t^{-1}) = ct^k\varphi(t)$, then the integral expression above gives the functional equation

$$M\Phi(k - s) = cM\Phi(s).$$

This will be our main source of analytic continuation and functional equations!

L-functions of modular forms

(I) For instance, take $f = \sum_{n \geq 1} a_n q^n \in S_k$. The ***L*-function of** f

$$L(s) = L(f, s) = \sum_{n \geq 1} \frac{a_n}{n^s}$$

and the **completed *L*-function of** f

$$\Lambda(f, s) := (2\pi)^{-s} \Gamma(s) L(f, s)$$

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are of fundamental importance.

(II) Note that by the above discussion $\Lambda(f,\bullet)$ is simply the Mellin transform of

$$\Phi(t) = f(it) = \sum_{n\geq 1} a_n e^{-2\pi n t}.$$

Since $f(-1/z) = z^k f(z)$, we have

$$\Phi(1/t) = i^k t^k \Phi(t) = (-1)^{k/2} t^k \Phi(t).$$

L-functions of modular forms

(I) We say that a function F is **EBV** if F is entire (i.e. holomorphic on $\mathbb{C}$) and bounded on any vertical strip $a \leq \mathrm{Re}(s) \leq b$. The previous discussion gives:

Theorem (Hecke) If $f \in S_k$ then $s \to \Lambda(f, s)$ extends to a EBV function satisfying the functional equation

$$\Lambda(k - s) = (-1)^{k/2} \Lambda(s).$$

Thus $L(f, \bullet)$ extends to an entire function, having (trivial) zeros at integers $n \leq 0$.

The values $L(f, n)$ for $1 \leq n \leq k - 1$ have very interesting arithmetic signification (at least for a class of modular forms we'll encounter later on, called eigenforms).

L-functions of modular forms

(I) Amazingly, the above properties of the L-function characterise elements of S_k.

Theorem (Hecke's converse theorem) Let $f = \sum_{n\geq 1} a_n q^n$ be a holomorphic function on $\mathscr{H}$ such that $a_n = O(n^v)$ for some $v > 0$. If

$$\Lambda(s) := (2\pi)^{-s}\Gamma(s)\sum_{n\geq 1}\frac{a_n}{n^s}$$

extends to an EBV function satisfying

$$\Lambda(k-s) = (-1)^{k/2}\Lambda(s),$$

then $f \in S_k$.

L-functions of modular forms

(I) Let $\Phi(t) = f(it)$, we need to show that $\Phi(1/t) = i^k t^k \Phi(t)$, since this implies $f(-1/z) = z^k f(z)$. As above, we have $\Lambda(s) = M\Phi(s)$.

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(II) The inversion formula for Mellin transforms gives for $t > 0, \sigma > 1 + v$

$$\Phi(t) = \frac{1}{2i\pi} \int_{\mathrm{Re}(s)=\sigma} \Lambda(s) t^{-s} ds$$

and similarly, using the functional equation of Λ we obtain

$$(it)^{-k}\Phi(1/t) = \frac{1}{2i\pi} \int_{\mathrm{Re}(s)=k-\sigma} \Lambda(s) t^{-s} ds.$$

(I) To show that these two integrals are the same, integrate the holomorphic map $F(s) = \Lambda(s)t^{-s}$ over the rectangle with vertices $(\sigma, T), (k - \sigma, T)$, $(k - \sigma, -T), (\sigma, -T)$ with T large enough. It suffices to prove that

$$\lim_{|T| \to \infty} \int_{k-\sigma}^{\sigma} F(u + iT) du = 0.$$

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$$\lim_{|T| \to \infty} \int_{k-\sigma}^{\sigma} F(u + iT)du = 0.$$

(II) This is an application of the Phragmen-Lindelf principle in complex analysis, which implies that $|\Lambda(s)| = O(1/|s|)$ uniformly in $k - \sigma \leq \mathrm{Re}(s) \leq \sigma$.

L-functions of modular forms

(I) Things are quite a bit more complicated if $N > 1$. The matrix $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ normalizes $\Gamma_0(N)$ and we obtain an operator W_N (called the **Fricke involution**) on $M_k(N)$, preserving $S_k(N)$ and satisfying

$$W_N f(z) = N^{-k/2} z^{-k} f(\frac{-1}{Nz}).$$

The same argument gives

Theorem If $f \in S_k(N)$, then

$$\Lambda(f,s) := N^{s/2}\pi^{-s}\Gamma(s)L(f,s)$$

extends to an EBV function satisfying

$$\Lambda(f,s) = i^k \Lambda(W_N f, k-s).$$

L-functions of modular forms

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(II) There is also an analogue of Hecke's converse theorem, but it is much more delicate to state and to prove. This is called Weil's converse theorem, and essentially characterises cusp forms in terms of L-functions of various twists

$$f \otimes \chi := \sum_{n \geq 1} a_n \chi(n) q^n$$

of f by Dirichlet characters χ mod D, for various D. The precise statement is very technical and skipped.

The Petersson inner product

(I) We have already seen that cuspidal automorphic forms of level Γ for $G = \mathbb{SL}_2(\mathbb{R})$ are in $L^2(\Gamma \backslash G)$, and that any $f \in S_k(\Gamma)$ (with $k \geq 1$) gives rise to a cuspidal automorphic form of level Γ for G.

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(II) Say Γ has finite index in $\Gamma(1)$ and, for simplicity, that $-1 \in \Gamma$. We get an embedding $S_k(\Gamma) \subset L^2(\Gamma\backslash G)$, inducing a Hilbert structure on the (finite-dimensional!) vector space $S_k(\Gamma)$. Unwinding definitions, we see that up to a constant this inner product can be expressed directly as

$$\langle f, g\rangle = \frac{1}{[\Gamma(1) : \Gamma]} \int_{\Gamma\backslash\mathscr{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^k d\mu(z).$$

It is independent of the choice of Γ for which both f, g are modular of level Γ and it will play a major role in the sequel!

The Petersson inner product

(I) More precisely, if F_f, F_g are the automorphic forms attached to $f, g \in S_k(\Gamma)$ (recall that $F_f(x) = f(x.i)\mu(x,i)^{-k}$)

$$\langle F_f, F_g \rangle_{L^2(\Gamma\backslash G)} = \int_{\Gamma\backslash G} F_f(x)\overline{F_g(x)}dx =$$

$$\int_{\Gamma\backslash G} f(x.i)\overline{g(x.i)}|\mu(x,i)|^{-2k}dx = \int_{\Gamma\backslash G} f(x.i)\overline{g(x.i)}\mathrm{Im}(x.i)^k dx$$

$$= \int_{\Gamma\backslash\mathscr{H}} f(z)\overline{g(z)}\mathrm{Im}(z)^k d\mu(z) = [\Gamma(1):\Gamma]\langle f, g\rangle.$$

The next-to-last equality follows by K-invariance on the right of the map $x \to f(x.i)\overline{g(x.i)}\mathrm{Im}(x.i)^k$.

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(II) The independence of $\langle f, g\rangle$ with respect to Γ (for which both f, g are modular) is then trivial.

The Rankin-Selberg method

(I) Let $f, g \in S_k(\Gamma)$ and let

$$L(f \otimes \bar{g}, s) = \zeta(2s - 2k + 2) \sum_{n \geq 1} \frac{a_n \overline{b_n}}{n^s}.$$

This is called the **Rankin-Selberg convolution of** $L(f, s)$ **and** $L(g, s)$. It is holomorphic in $\mathrm{Re}(s) > k + 1$ by the trivial bound.

Theorem (Rankin-Selberg) The series defining $L(f \otimes \bar{g}, s)$ converges absolutely if $\mathrm{Re}(s) > k$ and $L(f \otimes \bar{g}, \bullet)$ extends meromorphically to $\mathbb{C}$. This function may have a simple pole at $s = k$, where the residue is

$$\mathrm{Res}_{s=k} L(f \otimes \bar{g}, \bullet) = \frac{3}{\pi} \frac{(4\pi)^k}{(k-1)!} \langle f, g \rangle.$$

The Rankin-Selberg method

(I) The convergence in the theorem is highly nontrivial. Using a hard theorem of Landau from analytic number theory, Rankin deduced that for $f \in S_k(\Gamma_0(N))$ we have

$$\sum_{n \leq x} |a_n(f)|^2 = \frac{3}{\pi} \frac{(4\pi)^k}{(k-1)!} \langle f, f \rangle x^k + O(x^{k-\frac{2}{5}})$$

and

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and

$$a_n(f) = O(n^{\frac{k}{2} - \frac{1}{5}}).$$

(II) Again, we have a functional equation:

$$\Lambda(f \otimes \bar{g}, s) = \Lambda(f \otimes \bar{g}, 2k - 1 - s),$$

where

$$\Lambda(f \otimes \bar{g}, s) = (2\pi)^{-2s} \Gamma(s) \Gamma(s - k + 1) L(f \otimes \bar{g}, s)$$

The Rankin-Selberg method

(I) The proof of the theorem crucially relies on real analytic Eisenstein series (which had a different name in lecture 6)

$$E(s,z) = \frac{1}{2} \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^s}{|cz+d|^{2s}} = \frac{\zeta(2s)}{2} \sum_{\gcd(c,d)=1} \frac{y^s}{|cz+d|^{2s}}$$

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(II) The version at level $\Gamma := \Gamma_0(N)$ (and the cusp ∞) is

$$E_N(s,z) = \sum_{\gamma\in\Gamma_\infty\setminus\Gamma} \frac{y^s}{|cz+d|^{2s}} = \frac{1}{2} \sum_{\gcd(c,d)=1, N|c} \frac{y^s}{|cz+d|^{2s}}$$

The Mobius inversion formula allows one to relate this to $E(s,z)$.

The Rankin-Selberg method

(I) More precisely, using the Mobius function we can rewrite (details left as an exercise)

$$E_N(s,z) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\} \\ N|c}} \sum_{e|\gcd(c,d)} \mu(e) \frac{y^s}{|cz+d|^{2s}}$$

$$= \sum_{d|N} E(s, Nz/d)(N/d)^{-s} \sum_{\gcd(N,e)=d} \mu(e) e^{-2s} =$$

$$= \frac{1}{N^s \prod_{p|N}(1-p^{-2s})\zeta(2s)} \sum_{d|N} \frac{\mu(d)}{d^s} E(s, \frac{Nz}{d}).$$

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(II) The properties of $E(s,z)$ established in the previous lecture then ensure that E_N also extends meromorphically to $\mathbb{C}$.

The Rankin-Selberg method

(I) More precisely, letting

$$\Lambda(s) := \pi^{-s}\Gamma(s)\zeta(2s),$$

the map $s \to \Lambda(s)E_N(s)$ has a unique pole at 1 in the region $\mathrm{Re}(s) \geq 1/2$, this pole is simple with residue $\frac{1}{2[\Gamma(1):\Gamma]}$. Also, it satisfies a functional equation.

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(II) It follows that E_N has a single pole in $\mathrm{Re}(s) \geq 1/2$, at $s = 1$, and this pole is simple, with residue $\frac{1}{3\pi[\Gamma(1):\Gamma]}$. Everything in the theorem can be easily deduced from the following crucial identity, which holds for $\mathrm{Re}(s) > 1$:

$$\frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}[\Gamma(1):\Gamma_0(N)]}\sum_{n\geq 1}\frac{a_n\overline{b_n}}{n^s} = \langle E_N(s)f, g\rangle,$$

and the properties of $E_N(s)$ explained above.

(I) To prove the identity, we use as always the unfolding trick, using that $\varphi(z) = f(z)\overline{g(z)}\mathrm{Im}(z)^k$ is Γ-invariant

$$\langle E_N(s)f, g\rangle = \frac{1}{[\Gamma(1):\Gamma]}\int_{\Gamma\backslash\mathscr{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s \varphi(\gamma z)\, d\mu(z)$$

$$= \frac{1}{[\Gamma(1):\Gamma]}\int_{\Gamma_\infty\backslash\mathscr{H}} \mathrm{Im}(z)^s \varphi(z)\, d\mu(z) =$$

$$= \frac{1}{[\Gamma(1):\Gamma]}\int_0^\infty \left(\int_0^1 f(x+iy)\overline{g(x+iy)}\,dx\right) y^{k+s-1}\frac{dy}{y}.$$

The Rankin-Selberg method

(I) But a simple calculation shows that

$$\int_0^1 f(x+iy)\overline{g(x+iy)}dx = \sum_{n\geq 1} a_n \overline{b_n} e^{-4\pi ny}$$

and the result follows by standard properties of the Mellin transform. More precisely,

$$f(x+iy)\overline{g(x+iy)} = (\sum a_n e^{-2\pi ny} e^{2i\pi nx})(\sum \overline{b_m} e^{-2\pi my} e^{-2i\pi mx})$$

$$= \sum_{m,n} a_n \overline{b_m} e^{-2\pi(m+n)y} e^{2i\pi(n-m)x}$$

and the previous identity follows immediately.

Hecke operators

(I) Ramanujan conjectured the following amazing identity for $\Delta = \sum_{n\geq 1} \tau(n) q^n \in S_{12}$

$$L(\Delta, s) = \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}},$$

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(II) Hecke proved this result by introducing an extremely important class of operators on $M_k(N)$ and $S_k(N)$, called **Hecke operators**. Their definition looks very miraculous, and it is not easy at this stage to motivate it...

Hecke operators

(I) The spaces $M_k(N), S_k(N)$ turn out to be modules over the gigantic polynomial ring

$$\mathbb{T}_N = \mathbb{Z}[T_p | \, p \in \{2, 3, 5, 7, 11, ...\}]$$

in infinitely many variables T_p, indexed by prime numbers. Let $\mathbb{T}_N^{(N)}$ be the subring of $\mathbb{T}_N$ generated by T_p with $\gcd(p, N) = 1$.

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in infinitely many variables T_p, indexed by prime numbers. Let $\mathbb{T}_N^{(N)}$ be the subring of $\mathbb{T}_N$ generated by T_p with $\gcd(p, N) = 1$.

(II) We call $\mathbb{T}_N$ (resp. $\mathbb{T}_N^{(N)}$) the **universal Hecke algebra of level** N (resp. the same away from N).

Hecke operators

(I) What is this module structure? Brutally, we can define the p**th Hecke operator** T_p on $M_k(N)$ by

$$T_p(f)(z) = \frac{1}{p}\sum_{i=0}^{p-1} f\left(\frac{z+i}{p}\right) + \chi(p)p^{k-1}f(pz),$$

where $\chi(p) = 1$ if p is prime to N and $\chi(p) = 0$ if $p \mid N$. Be careful that T_p really depends on the level N since $\chi(p)$ depends on N.

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(II) It is absolutely not obvious that $T_p(f) \in M_k(N)$ and T_p sends $S_k(N)$ into $S_k(N)$, etc, see the slides at the end of the lecture for the proof! The various T_p's commute (this is very easy using the formula above).

Hecke operators

(I) One can define a Hecke operator T_n on $M_k(N)$ and $S_k(N)$ by

$$T_n(f)(z) = \frac{1}{n} \sum_{\substack{ad=n, a>0 \\ \gcd(a,N)=1}} a^k \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right).$$

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Again, it is by no means obvious that $T_n(f)$ is in $M_k(N)$ or $S_k(N)$.

(II) Let $a_m(f)$ be the mth Fourier coefficient in the q-expansion of f at ∞. A simple computation gives

$$a_m(T_n(f)) = \sum_{d|\gcd(m,n), \gcd(d,N)=1} d^{k-1} a_{\frac{mn}{d^2}}(f) = a_n(T_m(f)).$$

In particular we have the key formula $a_1(T_n(f)) = a_n(f)$ and

$$T_p(f) = \sum_{n \geq 0} a_{pn}(f) q^n + \chi(p) p^{k-1} \sum_{n \geq 0} a_n(f) q^{np}.$$

Hecke operators

(I) Using the above formula in terms of q-expansions, it is not difficult to check that

$$T_m T_n = \sum_{d | \gcd(m,n), \gcd(d,N)=1} d^{k-1} T_{\frac{mn}{d^2}}$$

in $\mathrm{End}_{\mathbb{C}}(M_k(N))$, in particular

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$$T_m T_n = T_{mn} \text{ if } \gcd(m,n) = 1.$$

(II) The previous identity is equivalent to the following beautiful equality of Dirichlet series with coefficients in $\mathrm{End}_{\mathbb{C}}(M_k(N))$

$$\sum_{n\geq 1} \frac{T_n}{n^s} = \prod_{p|N} \frac{1}{1 - T_p p^{-s}} \prod_{p\nmid N} (1 - T_p p^{-s} + p^{k-1-2s})^{-1}$$

$$= \prod_p (1 - T_p p^{-s} + \chi(p) p^{k-1-2s})^{-1}.$$

Eigenforms

(I) This identity is at the heart of all Euler factorizations of L-functions of "nice" modular forms.

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(II) We say that $f \in M_k(N)$ is a $\mathbb{T}_N$-**eigenform** if f is an eigenvector of all T_p. In this case f gives rise to a morphism of rings

$$\theta_f : \mathbb{T}_N \to \mathbb{C}, \;\; T(f) = \theta_f(T)f \text{ for } T \in \mathbb{T}.$$

An identical discussion applies to $\mathbb{T}_N^{(N)}$. Obviously any $\mathbb{T}_N$-eigenform is a $\mathbb{T}_N^{(N)}$-eigenform, but the converse fails badly when $N > 1$!

Eigenforms

(I) Note that if f is a $\mathbb{T}_N$-eigenform, then writing $T_n(f) = \lambda_n f$ we have

$$a_n(f) = a_1(T_n(f)) = \lambda_n a_1(f),$$

and thus $a_1(f) \neq 0$ (since $f \neq 0$) and we can normalise f such that $a_1(f) = 1$, in which case $T_n(f) = a_n f$ for all n. We say that f is a **normalized eigenform** in this case.

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(II) The Dirichlet series identity for the T_n's yields in this case

$$L(f,s) = \prod_p \frac{1}{1 - a_p p^{-s} + \chi(p) p^{k-1-2s}},$$

thus the L-function of a normalized $\mathbb{T}_N$-eigenform has a nice Euler factorization!

Eigenforms

(I) For instance, consider S_{12}. If $f \in S_{12}$, then $f/\Delta \in M_0 = \mathbb{C}$, thus S_{12} is one-dimensional and Δ is necessarily a normalized $\mathbb{T}_1$-eigenform and we obtain that $L(\Delta, s)$ has an Euler factorisation as conjectured by Ramanujan! In particular, if we write

$$q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n,$$

the sequence $\tau(n)$ is multiplicative and

$$\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1})$$

for all primes p and all $n \geq 1$.

Eigenforms

(I) Thus if $f \in S_k$ is a $\mathbb{T}_1$-eigenform, and if we factor

$$X^2 - a_p X + p^{k-1} = (X - \alpha_{f,p})(X - \beta_{f,p}),$$

then

$$L(f,s) = \prod_p \frac{1}{(1 - \alpha_{f,p} p^{-s})(1 - \beta_{f,p} p^{-s})}.$$

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(II) If f, g are eigenforms, a long but simple calculation shows that $L(f \otimes \bar{g}, s)$ also has an Euler factorization, with

$$L_p(f \otimes \bar{g}, s) = \frac{1}{(1 - \alpha_{f,p}\beta_{g,p} p^{-s})(1 - \alpha_{f,p}\alpha_{g,p} p^{-s})}$$

$$\bullet \frac{1}{(1 - \beta_{f,p}\alpha_{g,p} p^{-s})(1 - \beta_{f,p}\beta_{g,p} p^{-s})}.$$

Eigenforms

(I) A fundamental and quite subtle (**not** formal!) result is that Hecke operators interact well with the Petersson inner product:

Theorem For n prime to N the operator T_n is self-adjoint on $S_k(N)$, and so is W_N. For general n the adjoint of T_n is $W_N^{-1} T_n W_N$. In particular, if $\gcd(n, N) = 1$ then T_n and W_N commute.

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(II) The spectral theorem combined with the previous one show that $S_k(N)$ has an orthogonal basis consisting of $\mathbb{T}_N^{(N)}$ (but **not** $\mathbb{T}_N$)-eigenforms.

Atkin-Lehner theory

(I) Things are however much more subtle for $\mathbb{T}_N^{(N)}$-eigenforms: we can no longer assume that $a_1(f) \neq 0$, and we have no control at primes dividing N.

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(II) If f is a $\mathbb{T}_N^{(N)}$-eigenform with $a_1(f) = 0$, then as above we get $a_n(f) = 0$ for all n prime to N. The next result is the heart of the Atkin-Lehner theory

Theorem (Atkin-Lehner's main lemma) If $f \in S_k(N)$ satisfies $a_n(f) = 0$ for all n prime to a given integer D, then $f(z) = \sum_{p|N} g_p(pz)$ for some $g_p \in S_k(N/p)$.

The converse is trivial! The classical proof is not very enlightening (but only 4 pages long!), while the automorphic proof, while conceptual and enlightening, is quite long.

Atkin-Lehner theory

(I) For each pair (d, M) of positive integers with $dM \mid N$ we have an injective morphism of $\mathbb{T}_N^{(N)}$-modules

$$\iota^* = \iota^*_{d,M} : S_k(M) \to S_k(N),\ f \to (z \to f(dz)).$$

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(II) In particular, if $dM \mid N$ and $f \in S_k(M)$ is a $\mathbb{T}_M^{(M)}$-eigenform, then $z \to f(dz)$ is a $\mathbb{T}_N^{(N)}$-eigenform having the same T_p-eigenvalue as f for all p prime to N! Thus $\mathbb{T}_N^{(N)}$-eigenspaces can have big dimension, contrary to $\mathbb{T}_N$-eigenspaces.

Atkin-Lehner theory

(I) The old and new subspaces of $S_k(N)$ are defined by

$$S_k(N)^{\mathrm{old}} = \sum_{dM|N, M \neq N} \iota_{d,M}^*(S_k(M)), \quad S_k(N)^{\mathrm{new}} = (S_k(N)^{\mathrm{old}})^{\perp}.$$

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(II) One can check by direct computation that $S_k(N)^{\mathrm{old}}$ is stable under $\mathbb{T}_N$ and by the Fricke involution W_N. It follows that $S_k(N)^{\mathrm{new}}$ is also stable under $\mathbb{T}_N$ and W_N.

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(III) The previous deep theorem ensures that whenever $f \in S_k(N)^{\mathrm{new}}$ is a $\mathbb{T}^{(N)}$-eigenform, we necessarily have $a_1(f) \neq 0$, thus we can normalise $a_1(f) = 1$. We call such f a **newform or primitive form**.

Atkin-Lehner theory

(I) We can now state the main results of Atkin-Lehner theory.

Theorem (Atkin-Lehner)

a) (multiplicity one) $\mathbb{T}_N^{(N)}$-eigenspaces in $S_k(N)^{\mathrm{new}}$ are one-dimensional and generated by primitive forms.

b) Each primitive form is a $\mathbb{T}_N$-eigenform and an eigenvector of W_N, thus $L(f,s)$ has an Euler factorization and $\Lambda(f,s)$ has a nice functional equation.

c) $S_k(N)^{\mathrm{new}}$ has a basis consisting of primitive forms.

One also gets a decomposition of $\mathbb{T}_N^{(N)}$-modules

$$S_k(N) = \bigoplus_{dM|N} \iota_{d,M}^*(S_k(M)^{\mathrm{new}}) \simeq \bigoplus_{dM|N} S_k(M)^{\mathrm{new}}.$$

Atkin-Lehner theory

(I) The proof of the following deep and beautiful **strong multiplicity one** theorem crucially uses L-functions and estimates on the growth of the coefficients of cusp forms obtained via Rankin-Selberg:

Theorem (Atkin-Lehner) If $f \in S_k(M), g \in S_k(N)$ are primitive forms with $a_p(f) = a_p(g)$ for all but finitely many primes p, then $M = N$ and $f = g$.

Two amazing theorems

(I) Here is the first amazing theorem:

Theorem (Shimura) If $f = \sum_{n\geq 1} a_n q^n \in S_k(N)$ is primitive, then

$$\mathbb{Q}(f) := \mathbb{Q}(a_1, a_2, \ldots)$$

is a totally real number field in which a_n are algebraic integers. Moreover, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ the holomorphic function $f^\sigma = \sum_{n\geq 1} \sigma(a_n) q^n$ is a primitive form in $S_k(N)$.

This is not too hard for $N = 1$: in this case if $d = \dim M_k$ one easily checks that $\Delta^i E_4^{3(d-1-i)} E_{k-12d+12}$ for $0 \leq i < d$ form a basis of M_k and these forms have integer Fourier coefficients. Using them, one checks that the subspace $M_k(\mathbb{Z})$ of M_k of forms with integer q-expansions is a lattice in M_k, clearly stable under Hecke operators, and the result follows easily.

Two amazing theorems

(I) And here is one jewel of mathematics:

Theorem (Shimura, Deligne, Deligne-Serre, Ribet) Let $f = \sum_{n\geq 1} a_n q^n \in S_k(N)$ be a primitive form and let λ be a place of $K = \mathbb{Q}_f$ above some prime number ℓ. There is a continuous irreducible representation

$$\rho_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{GL}_2(K_\lambda)$$

such that for all primes p not dividing ℓN the representation $\rho_{f,\lambda}$ is trivial on the inertia group at p and

$$\det(X - \rho_{f,\lambda}(\mathrm{Frob}_p)) = X^2 - a_p X + p^{k-1}.$$

Moreover, the Ramanujan-Petersson conjecture holds:

$$|a_p| \leq 2p^{\frac{k-1}{2}}.$$

Hecke operators: where from?

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(III) Each $g \in G$ gives rise to an operator

$$[\Gamma g\Gamma] : M^\Gamma \to M^\Gamma,\ m.[\Gamma g\Gamma] = \sum_{x\in\Gamma\backslash\Gamma g\Gamma} m.x,$$

depending only on $\Gamma g\Gamma$.

Hecke operators: where from?

(I) Let $\mathbb{T}(G, \Gamma) = A[\Gamma \backslash G / \Gamma]$ be the free A-module on double classes $\Gamma g \Gamma$ with $g \in G$. We have a natural bijection

$$\mathrm{Hom}_{A[G]}(A[\Gamma \backslash G], M) \simeq M^{\Gamma},$$

in particular taking $M = A[\Gamma \backslash G]$ with the obvious action of G

$$\mathrm{End}_{A[G]}(A[\Gamma \backslash G]) \simeq A[\Gamma \backslash G]^{\Gamma} \simeq A[\Gamma \backslash G / \Gamma].$$

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(II) Since $\mathrm{End}_{A[G]}(A[\Gamma\backslash G])$ has a natural ring structure, we get one on $\mathbb{T}(G,\Gamma)$, and a $\mathbb{T}(G,\Gamma)$-module structure on M^{Γ}. This module structure is induced by the operators $[\Gamma g\Gamma]$.

Hecke operators: where from?

(I) An example of such a situation is $G = \mathbb{GL}_2(\mathbb{Q})^+$ and Γ any finite index subgroup of $\Gamma(1)$. We can take $A = \mathbb{C}$ (or $\mathbb{Z}$) and (for a fixed k)

$$M = \bigcup_{\Gamma} M_k(\Gamma)$$

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(II) The group G acts (good exercise: why?) on M by

$$f|_k g(z) := \det(g)^{k-1} f(gz)\mu(g,z)^{-k}$$

and for any finite index subgroup Γ of $\Gamma(1)$ we have

$$M_k(\Gamma) = M^{\Gamma} = \{m \in M |\, m.\gamma = m,\ \gamma \in \Gamma\}.$$

Hecke operators: where from?

(I) Consider now $N \geq 1$, $\Gamma = \Gamma_0(N)$ and the set

$$H_n = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_2(\mathbb{Z}) |\, ad - bc = n,\ \gcd(a, N) = 1, N \mid c\}.$$

Since H_n is stable under left and right multiplication by Γ, it is a disjoint union of (finitely many, cf. below) double classes, say $H_n = \coprod_i \Gamma\alpha_i\Gamma$, and thus we have an operator

$$T_n = \sum_i [\Gamma\alpha_i\Gamma] : M_k(N) \to M_k(N).$$

Concretely,

$$T_n(f) = \sum_{\gamma \in \Gamma \backslash H_n} f|_k\gamma.$$

Hecke operators: where from?

(I) But elementary divisors theorem easily implies that

$$H_n = \coprod_{\substack{ad=n, a>0 \\ \gcd(a,N)=1}} \coprod_{b=0}^{d-1} \Gamma \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

hence

$$T_n(f) = \sum_{\substack{ad=n, a>0 \\ \gcd(a,N)=1}} \sum_{b=0}^{d-1} f|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

which matches the definition we initially took for T_n. Now we finally know that T_n sends modular forms to modular forms!

Hecke operators: where from?

(I) Using this description, one can also prove that Hecke operators are self-adjoint, as follows (the proof is quite tricky). If $\alpha \in G := \mathbb{GL}_2(\mathbb{Q})^+$ let $\alpha^* = \det(\alpha)\alpha^{-1}$. In a first step one proves that $\langle f|_k\alpha, g\rangle = \langle f, g|_k\alpha^*\rangle$ by a change of variable.

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(II) Using also the independence of the Petersson inner product with respect to the level, it follows that if $\Gamma\alpha\Gamma = \coprod \Gamma\alpha_i$ then

$$\langle f|[\Gamma\alpha\Gamma], g\rangle = \langle f, \sum g|_k\alpha_i^*\rangle$$

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(III) The tricky thing is to prove that one can choose the α_i's such that $\Gamma\alpha^*\Gamma = \coprod \Gamma\alpha_i^*$, as then one can repeat the above argument and get

$$[\Gamma\alpha\Gamma]^* = [\Gamma\alpha^*\Gamma].$$

This easily implies the result, coming back to the description of T_n in terms of double classes.

Hecke operators: where from?

(I) Elementary manipulations reduce this to showing that one can choose α_i such that $\Gamma\alpha\Gamma = \coprod \Gamma\alpha_i = \coprod \alpha_i\Gamma$, and simple group theory reduces this further to

$$|\Gamma\backslash\Gamma\alpha\Gamma| = |\Gamma\alpha\Gamma/\Gamma|.$$

This in turn reduces to checking that $\Gamma \cap \alpha^{-1}\Gamma\alpha$ and $\Gamma \cap (\alpha^*)^{-1}\Gamma\alpha^*$ have the same co-volume, which is easy enough.