Lecture 6: $\mathbb{SL}_2(\mathbb{R})$, almost game over

Gabriel Dospinescu

CNRS, ENS Lyon

The Selberg trace formula for compact hyperbolic curves

(I) Let us recall the setup and main results of the previous lecture. We start with a cocompact lattice Γ in G such that $|\mathrm{tr}(\gamma)| > 2$ for all $\gamma \in \Gamma \setminus \{\pm 1\}$. This is equivalent to saying that Γ has no nontrivial torsion elements except maybe -1 (exercise). For simplicity we assume that $-1 \notin \Gamma$.

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(II) Consider the compact hyperbolic curve $X = \Gamma \backslash \mathscr{H}$. We saw that $L^2(X)$ has an ON-basis consisting of eigenfunctions of the hyperbolic Laplacian Δ. Order the eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$$

We saw that each eigenvalue appears with finite multiplicity.

The Selberg trace formula for compact hyperbolic curves

(I) We also proved a general "abstract" trace formula for compact quotients, which in our case becomes, for $f \in C_c^\infty(G)$

$$\sum_{\pi \in \hat{G}} m(\pi)\mathrm{tr}(\pi(f)) = \sum_{\gamma \in \{\Gamma\}} \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)dx,$$

where

$$L^2(\Gamma \backslash G) = \widehat{\bigoplus_{\pi \in \hat{G}}} \pi^{\oplus m(\pi)}$$

is the GGPS decomposition, G_γ and Γ_γ are the centralizers of γ in G and Γ, $\{\Gamma\}$ is the set of Γ-conjugacy classes in Γ and $\mathrm{tr}(\pi(f))$ is the trace of the operator T_f on π.

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(II) We will pick $f \in \mathrm{Sph} := C_c^\infty(G//K)$. Then T_f sends π into π^K, thus we can restrict to $\pi \in \hat{G}^{\mathrm{sph}}$, where

$$\hat{G}^{\mathrm{sph}} = \{\pi \in \hat{G} | \, \pi^K \neq 0\} = \{\pi_s | \, s \in i\mathbb{R}^+ \cup (0,1)\}.$$

The Selberg trace formula for compact hyperbolic curves

(I) We proved last time that $m(\pi_s)$ is the dimension of the space of $f \in C^\infty(X)$ with $\Delta f = \frac{1-s^2}{4} f$. Thus if we write $\lambda_j = \frac{1}{4} + r_j^2$ with $r_j \in \mathbb{R}^+ \cup \frac{1}{2i}(0,1)$, then $m(\pi_s)$ is the number of j for which $r_j = \frac{s}{2i}$.

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(II) Recall that Sph acts on the one-dimensional space π_s^K by a character $\chi_{\pi_s} : \mathrm{Sph} \to \mathbb{C}$ and we showed last time that

$$\chi_{\pi_s}(f) = \hat{g}(\frac{s}{2i}), \hat{\varphi}(x) := \int_{\mathbb{R}} \varphi(t) e^{ixt} dt,$$

with

$$g(u) = HC(f)(u) = e^{u/2} \int_{\mathbb{R}} f(\begin{pmatrix} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}) dx$$
$$= \int_{\mathbb{R}} f(\begin{pmatrix} e^{u/2} & x \\ 0 & e^{-u/2} \end{pmatrix}) dx$$

the Harish-Chandra (or Selberg) transform of f.

The Selberg trace formula for compact hyperbolic curves

(I) Since T_f sends π_s into π_s^K, which is a line, it is clear that

$$\mathrm{tr}(\pi_s(f)) = \chi_{\pi_s}(f).$$

It follows that the spectral part of the abstract trace formula is

$$\sum_s \hat{g}(\frac{s}{2i}) \sum_{r_j = \frac{s}{2i}} 1 = \sum_j \hat{g}(r_j).$$

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(II) We also saw that sending f to $g = HC(f)$ yields an isomorphism (of vector spaces) $\mathrm{Sph} \simeq C_c^\infty(\mathbb{R})^{\mathrm{even}}$ and that we have the "Fourier inversion formula"
$$f(1) = \frac{1}{2\pi} \int_0^\infty x\hat{g}(x) \tanh(\pi x) dx.$$
In particular the term corresponding to the class of 1 in the geometric side of the trace formula is
$$\mathrm{vol}(\Gamma \backslash G) f(1) = \frac{\mathrm{area}(X)}{2\pi} \int_0^\infty x\hat{g}(x) \tanh(\pi x) dx.$$

The Selberg trace formula for compact hyperbolic curves

(I) It remains to understand the contribution of the other γ. Fix $\gamma \in \Gamma \setminus \{\pm 1\}$. Then γ is conjugated to $\pm a_\gamma = \pm \begin{pmatrix} e^{l(\gamma)/2} & 0 \\ 0 & e^{-l(\gamma)/2} \end{pmatrix}$ with

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$$l(\gamma) = 2\mathrm{arccosh}(\frac{|\mathrm{tr}(\gamma)|}{2}).$$

(II) One easily checks that any matrix conjugating γ to $\pm a_\gamma$ also conjugates G_γ to $\pm A$ (which is the centralizer of a_γ). We need to understand Γ_γ. Note that if $\gamma' \in \Gamma_\gamma$, then γ, γ' are simultaneously conjugate to $\pm a_\gamma$ and $\pm a_{\gamma'}$, thus $\gamma\gamma'$ is conjugate to $\pm \begin{pmatrix} e^{(l(\gamma)+l(\gamma'))/2} & 0 \\ 0 & e^{-(l(\gamma)+l(\gamma'))/2} \end{pmatrix}$ and so

$$l(\gamma\gamma') = l(\gamma) + l(\gamma').$$

The Selberg trace formula for compact hyperbolic curves

(I) Let us assume for simplicity that $-1 \notin \Gamma$. It follows that $l : \Gamma_\gamma \to \mathbb{R}$ is a morphism of groups, which is trivially injective, thus Γ_γ is cyclic, generated by some γ_0. One easily checks that the various γ_0^n (with $n \geq 1$) are pairwise not Γ-conjugate.

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(II) Now $\mathrm{vol}(\Gamma_\gamma \backslash G_\gamma)$ is easily computable in terms of $l_0 = l(\gamma_0)$, namely an absolute constant (exercise: which one?) times l_0.

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(II) Now $\mathrm{vol}(\Gamma_\gamma \backslash G_\gamma)$ is easily computable in terms of $l_0 = l(\gamma_0)$, namely an absolute constant (exercise: which one?) times l_0.

(III) To compute $\int_{G_\gamma \backslash G} f(x^{-1}\gamma x)$ by a change of variable we reduce to computing $\int_{\pm A \backslash G} f(x^{-1} a_\gamma x)$.

The Selberg trace formula for compact hyperbolic curves

(I) This is also (recall that f is bi-K-invariant)

$$\frac{1}{2}\int_{A\backslash G} f(x^{-1}a_\gamma x)dx = \frac{1}{2}\int_K\int_N f((nk)^{-1}a_\gamma nk)dndk$$

$$= \frac{1}{2}\int_N f(n^{-1}a_\gamma n)dn = \frac{1}{2}\int_{\mathbb{R}} f(\begin{pmatrix} e^{l(\gamma)/2} & (e^{l(\gamma)/2}-e^{-l(\gamma)/2})x \\ 0 & e^{-l(\gamma)/2}\end{pmatrix} dx$$

$$= \frac{1}{2(e^{l(\gamma)/2}-e^{-l(\gamma)/2})}\int_{\mathbb{R}} f(\begin{pmatrix} e^{l(\gamma)/2} & x \\ 0 & e^{-l(\gamma)/2}\end{pmatrix} dx$$

$$= \frac{1}{2(e^{l(\gamma)/2}-e^{-l(\gamma)/2})} g(l(\gamma)).$$

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$$= \frac{1}{2(e^{l(\gamma)/2} - e^{-l(\gamma)/2})} g(l(\gamma)).$$

(II) Combining all these computations yields the Selberg trace formula for X, as stated in the previous lecture.

The finiteness theorem

(I) Let Γ be a lattice in $G = \mathbb{SL}_2(\mathbb{R})$ and let

$$H = L^2(\Gamma\backslash G).$$

Let $H_{\mathrm{cusp}} = L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ be the cuspidal subspace of H.

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Theorem H_{cusp} is a closed subspace of H.

(II) Say $f_n \in H_{\text{cusp}}$ converge in H to some f. Fix $P \in CP(\Gamma)$, we want to prove that $f_P = 0$. Since f_P is by design left N-invariant, it suffices to check that $\int_{N \backslash G} f_P(g)\alpha(g)dg = 0$ for all test functions $\alpha \in C_c^\infty(N \backslash G)$. Since we know that this holds for f_n instead of f, it suffices to show that for a fixed α the linear form $f \to \int_{N \backslash G} f_P(g)\alpha(g)dg$ is continuous.

The finiteness theorem

(I) But

$$\int_{N\backslash G} f_P(g)\alpha(g)dg = \int_{N\backslash G} \alpha(g) \int_{\Gamma_N\backslash N} f(ng)dn =$$

$$\int_{\Gamma_N\backslash G} \alpha(g)f(g)dg = \int_{\Gamma\backslash G} (\sum_{\gamma\in\Gamma_N\backslash\Gamma} \alpha(\gamma g))f(g)dg.$$

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(II) So it suffices to check that $F(g) := \sum_{\gamma\in\Gamma_N\backslash\Gamma} \alpha(\gamma g)$ is bounded. Since α has compact support modulo N and N is compact modulo Γ_N, there is a compact C such that $\mathrm{Supp}(\alpha) \subset \Gamma_N C$.

The finiteness theorem

(I) The number of $\gamma \in \Gamma_N \backslash \Gamma$ with $\alpha(\gamma g) \neq 0$ is at most the number of $\gamma \in \Gamma$ with $\gamma g \in C$. Now $\gamma g \in C$ and $\gamma' g \in C$ forces $\gamma' \gamma^{-1} \in CC^{-1}$ and since CC^{-1} is compact, it follows that there is a constant c such that for each g there are at most c nonzero terms in the sum defining F, and since α is bounded, we are done.

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(II) Next, we parameterize $\hat{K}$ by $\mathbb{Z}$ via $m \to \chi_m$ with

$$\chi_m(\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}) = e^{imt}.$$

If $V \in \mathrm{Rep}(K)$ let

$$V_m = \{v \in V |\, k.v = \chi_m(k)v, k \in K\}$$

be the subspace of vectors of K-type m. By Peter-Weyl, if V is unitary then

$$V = \widehat{\oplus_{m \in \mathbb{Z}}} V_m.$$

The finiteness theorem

(I) Fix an integer m in the sequel. If $s \in \mathbb{C}$ let

$$A(\Gamma)^s_m = \{f \in A(\Gamma)_m | \mathscr{C} f = \frac{1-s^2}{2} f\}.$$

Theorem a) The space $A(\Gamma)^s_m$ is finite dimensional.

b) $H_{\mathrm{cusp,m}}$ has an orthonormal basis consisting of smooth vectors which are eigenvectors of $\mathscr{C}$. Each such eigenvector is in $A(\Gamma)^s_m$ for some s, thus eigenspaces of $\mathscr{C}$ on $H^{\infty}_{\mathrm{cusp,m}}$ are finite dimensional.

It follows from b) that $A_{\mathrm{cusp}}(\Gamma)$ is dense in H_{cusp}, which is not at all trivial a priori!

The finiteness theorem

(I) The proof will keep us busy for a while. Recall that any $f \in A_{\mathrm{cusp}}(\Gamma)$ is rapidly decreasing on Siegel sets, in particular bounded on any Siegel set. Since finitely many Siegel sets cover $\Gamma\backslash G$, f is bounded on G and so

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(II) The next beautiful result will be the first key ingredient in the proof of part a) of the previous theorem:

Theorem (Godement's lemma) Let X be a finite measure space and $V \subset L^2(X)$ a closed subspace such that $V \subset L^{\infty}(X)$. Then V is finite dimensional.

The finiteness theorem

(I) The proof is extremely beautiful. First, it is not difficult to see that V is closed in $L^\infty(X)$. Next, the identity map $(V, ||\bullet||_\infty) \to (V, ||\bullet||_2)$ is clearly continuous, linear and bijective between the two Banach spaces, thus (by the open mapping theorem) it is a homeomorphism. Thus there is c such that

$$||f||_\infty \leq c||f||_2,\ f \in V.$$

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(II) Now let $f_1, \ldots, f_n \in V$ be an orthonormal family. We will show that n is bounded, which is enough to conclude. Pick a dense countable set $S \subset \mathbb{C}$ and $a_1, \ldots, a_n \in S$. Letting $f = \sum a_i f_i$, we obtain

$$|\sum_{i=1}^{n} a_i f_i(x)| \leq c\sqrt{\sum_{i=1}^{n} |a_i|^2}$$

for almost all x.

The finiteness theorem

(I) Since S is countable, we deduce that for almost all x, the previous inequality holds for any $a_1, \ldots, a_n \in S$ and then, by continuity and density, for all $a_i \in \mathbb{C}$, in particular for $a_i = \overline{f_i(x)}$. Thus for almost all x we have

$$\sum_{i=1}^{n} |f_i(x)|^2 \leq c^2.$$

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$$\sum_{i=1}^{n} |f_i(x)|^2 \leq c^2.$$

(II) Integrate this over X to get

$$n = \int_X (\sum_{i=1}^{n} |f_i(x)|^2) dx \leq c^2 \int_X dx < \infty.$$

This finishes the proof.

The finiteness theorem

(I) Next, we prove that $A_{\mathrm{cusp}}(\Gamma)^s_m$ is closed in H. Combined with the previous observations and Godement's lemma, this will imply that it is finite dimensional. Say $f_n \in A_{\mathrm{cusp}}(\Gamma)^s_m$ converge in H to some f. It suffices to show that $f \in A(\Gamma)^s_m$, since we have already seen that H_{cusp} is closed in H (and $f_n \in H_{\mathrm{cusp}}$).

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(II) It is easy to see that f must be of K-type m and left Γ-invariant. Next, we prove that $\mathscr{C} f = \frac{s^2-1}{2} f$ in the sense of distributions. Since f is left Γ-invariant, it suffices to see that

$$\int_{\Gamma\backslash G} (\mathscr{C} - \frac{s^2-1}{2})\alpha(x) f(x) dx = 0$$

for all $\alpha \in C_c^\infty(\Gamma\backslash G)$. This equality holds with f_n instead of f and we can pass to the limit in L^2 sense by Cauchy-Schwarz (note that $(\mathscr{C} - \frac{s^2-1}{2})\alpha$ has compact support, thus it belongs to H).

The finiteness theorem

(I) At this point we can invoke the elliptic regularity and harmonicity theorem to deduce that f is smooth and $f = f * \alpha$ for some $\alpha \in C_c^\infty(G)$. But we saw while proving the GGPS theorem that there is c_α such that $|f * \alpha|_\infty \leq c_\alpha ||f||_2$ for any $f \in H_{\text{cusp}}$, in particular $f = f * \alpha$ is bounded, thus of moderate growth. We finally conclude that $f \in A(\Gamma)_m^s$.

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(II) Now we know that $A_{\mathrm{cusp}}(\Gamma)_m^s$ is finite dimensional. To go from here to $A(\Gamma)_m^s$ we need to understand constant terms at various cusps. Let $P_1, ..., P_l$ be a set of representatives for $\Gamma \backslash CP(\Gamma)$ and look at the map

$$\varphi : A(\Gamma)_m^s \to \prod_{i=1}^{l} \mathrm{Fct}(G),\ \varphi(f) = (f_{P_1}, ..., f_{P_l}).$$

The finiteness theorem

(I) The kernel of φ is $A_{\mathrm{cusp}}(\Gamma)^s_m$, so it suffices to check that its image is finite dimensional. Thus we are reduced to checking that the image of $f \to f_{P_i}$ is finite dimensional for each i. Now if P_i has unipotent radical N_i and A-component A_i, then f_{P_i} is left N_i-invariant and of K-type m, thus (by the Iwasawa decomposition) it is completely determined by its restriction to A_i. The relation $\mathscr{C} f = \frac{s^2-1}{2} f$ passes to f_{P_i} and a direct computation shows that this yields a linear differential equation with constant terms, of second order, satisfied by $f_{P_i}|_{A_i}$. The space of solutions of such an equation is finite dimensional, so we are done.

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(II) We now move to part b) of the theorem. We use the GGPS decomposition

$$H_{\text{cusp}} = \widehat{\bigoplus_{\pi \in \hat{G}}} \pi^{m(\pi)}.$$

The finiteness theorem

(I) Pass to vectors of K-type m to get

$$H_{\mathrm{cusp,m}} = \widehat{\bigoplus_{\pi \in \hat{G}, \pi_m \neq 0}} \pi_m^{m(\pi)}$$

All $\pi \in \hat{G}$ are admissible (we will see a proof in great generality later on, but it also follows from the classification), thus π_m are finite dimensional vector spaces contained in π^∞. Since π is unitary, elements of $\mathfrak{g}$ act by anti-symmetric operators on π^∞, thus $\mathscr{C}$ is self-adjoint on π_m, and thus diagonalizable in an orthonormal basis (to be fair, π_m is actually one dimensional if nonzero, but this is very specific to our G...). We deduce from here that $H_{\mathrm{cusp,m}}$ has an orthonormal basis of $\mathscr{C}$-eigenvectors.

The finiteness theorem

(I) We still need to show that if $f \in H^{\infty}_{\mathrm{cusp,m}}$ is an eigenvector for $\mathscr{C}$, then it is in $A(\Gamma)$. Of course, the problem is the moderate growth. But by harmonicity $f = f * \alpha$ for some test function α and, while proving the GGPS theorem, we saw that $f * \alpha$ is bounded, so we are done.

The finiteness theorem

(I) We note that it follows from the classification theorem that for each $\lambda \in \mathbb{C}$ there are at most two $\pi \in \hat{G}$ such that $\mathscr{C} = \lambda$ on π^∞. This combined with the previous discussion (based on the GGPS decomposition) yields an alternate proof that $A_{\mathrm{cusp}}(\Gamma)^s_m$ is finite dimensional. This kind of argument extends as well to other groups (so does the first), but then rests on a very difficult theorem of Harish-Chandra. The proof based on Godement's lemma avoids that deep result.

Modular forms and automorphic forms

(I) We want to relate modular forms and automorphic forms (something that we should have done long time ago...). The recipe is surprisingly simple. For any function f on $\mathscr{H}$ and any integer m we can lift f to a function F_f on G (depending on m as well)

$$F_f(g) = (f|_m g)(i) = f(g.i)\mu(g,i)^{-m},$$

where $\mu(g,z) = cz + d$ is the usual cocycle.

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where $\mu(g,z) = cz + d$ is the usual cocycle.

(II) Then one easily checks that $F_f(gk) = \chi_m(k)F_f(g)$ for $k \in K, g \in G$, and $F_{f|_m g}(x) = F_f(gx)$ for all $g, x \in G$. Moreover, $f \to F_f$ is injective since we can recover f from F_f by the simple but crucial formula

$$F_f(n_x a_y) = y^{m/2} f(x+iy),\ n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},\ a_y = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$$

So $f|_m\gamma = f$ for all $\gamma \in \Gamma$ if and only if F_f is left Γ-invariant.

Modular forms and automorphic forms

(I) Let

$$X_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \in \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}.$$

Theorem The map $f \to F_f$ induces an isomorphism

$$M_m(\Gamma) \simeq A(\Gamma)_m^{X_- = 0} := \{F \in A(\Gamma)_m | \, X_- F = 0\}$$

and for any $F \in A(\Gamma)_m^{X_- = 0}$ we have $\mathscr{C} F = (\frac{m^2}{2} - m)F$. Moreover $f \in M_m(\Gamma)$ is in $S_m(\Gamma)$ if and only if $F_f \in A_{\mathrm{cusp}}(\Gamma)_m$, thus

$$S_m(\Gamma) \simeq A_{\mathrm{cusp}}(\Gamma)_m^{X_- = 0}.$$

The differential equation $X_- F_f = 0$ is an incarnation of the Cauchy-Riemann equation for holomorphic functions.

(I) To avoid horrible computations it is better to compute using the basis of $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ given by

$$X_{\pm} = \frac{1}{2}\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}, \ H = -iW,$$

where $W = e - f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ satisfies $e^{tW} = r_t$. We have

$$[H, X_{\pm}] = \pm 2X_{\pm}, \ [X_+, X_-] = H$$

and

$$\mathscr{C} = \frac{H^2 + 2X_+X_- + X_-X_+}{2} = \frac{H^2 - 2H}{2} + 2X_+X_-.$$

Classical modular forms

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(II) The holomorphy of f is equivalent to $X_- F_f = 0$, thanks to the following identity (where $z = x + iy$)

$$(X_- F_f)(n_x a_y r_\theta) = -ie^{i(m-2)\theta} y^{1+\frac{m}{2}} (\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) f(z) \quad (1).$$

(I) To prove this, we first get rid of r_θ: using the relations $r_\theta X_- r_\theta^{-1} = e^{-2i\theta} X_-$ and $F_f(gr_\theta) = e^{im\theta} F_f(g)$, we easily obtain

$$(X_- F_f)(ur_\theta) = e^{i(m-2)\theta}(X_- F_f)(u),$$

thus we may assume that $\theta = 0$.

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$$(X_- F_f)(ur_\theta) = e^{i(m-2)\theta}(X_- F_f)(u),$$

thus we may assume that $\theta = 0$.

(II) Next, we decompose

$$X_- = -\frac{i}{2} W - ie + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $e^{tW} = r_t$ and $F_f(gr_t) = e^{imt} F_f(g)$, we easily obtain $HF_f = mF_f$.

Classical modular forms

(I) Since $n_x a_y \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = n_x a_{ye^{2t}}$, we obtain

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F_f(n_x a_y) = \frac{d}{dt}|_{t=0} F_f(n_x a_{ye^{2t}}) dt =$$

$$\frac{d}{dt}|_{t=0} (ye^{2t})^{m/2} f(x + ye^{2t} i) = y^{m/2} (mf(z) + 2y \frac{\partial f}{\partial y}(z)).$$

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(II) Similarly, using $n_x a_y n_t = n_{x+ty} a_y$, we obtain

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Combining these formulae yields (1).

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Combining these formulae yields (1).

(III) Finally, the formula for $\mathscr{C} F_f$ follows immediately from $HF_f = mF_f$, $X_- F_f = 0$ and

$$\mathscr{C} = \frac{H^2 - 2H}{2} + 2X_+ X_-.$$

Classical modular forms

(I) It remains to prove that f is holomorphic at cusps if and only if F_f has moderate growth. The moderate growth condition for F_f can be tested on Siegel sets at the various $P \in CP(\Gamma)$, thus it suffices to prove that for a fixed $P \in CP(\Gamma)$, f is holomorphic at the fixed point $z \in \partial\mathscr{H}$ of P if and only if F_f has moderate growth on a Siegel set Σ at z.

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(II) Conjugating everything, we may assume that $z = \infty$, thus $P = B$ and $\Gamma_\infty = \begin{pmatrix} 1 & h\mathbb{Z} \\ 0 & 1 \end{pmatrix}$ with $h > 0$, and we may take $\Sigma = \begin{pmatrix} 1 & [-c, c] \\ 0 & 1 \end{pmatrix} A_t K$ for some $c, t > 0$. Consider the q-expansion of f at ∞

$$f(z) = \sum_{n \in \mathbb{Z}} a_n e^{2i\pi nz/h}.$$

Classical modular forms

(I) Now, for $x \in [-c, c]$, $y > t$ and $k \in K$ we have (with $z = x + iy$)

$$|F_f(n_x a_y k)| = |F_f(n_x a_y)| = |y^{k/2} f(z)|$$

and $||n_x a_y k||$ behaves like $y^{1/2}$ on Σ. Thus we are reduced to showing the equivalence between:
- $a_n = 0$ for $n < 0$
- there are $M, C > 0$ such that $|f(z)| \leq C y^M$ for $y > t$ and $x \in [-c, c]$.

This is an elementary exercise. We also leave as an exercise the fact that f is cuspidal if and only if F_f is so.

Classical modular forms

(I) One can use the previous theorem and extra work to get the following very beautiful result:

Theorem (Gelfand, Graev, Piatetski-Shapiro) For any $m \geq 2$ there is a natural isomorphism

$$\mathrm{Hom}_G(DS_m^-, H_{\mathrm{cusp}}) \simeq S_m(\Gamma).$$

Thus $\dim S_m(\Gamma)$ is the multiplicity of DS_m^- in the GGPS decomposition of H_{cusp}.

Note that this is an analogue for DS_m^- of the equality

$$m(\pi_s) = \dim\{f \in C^\infty(X) | \mathscr{C} f = \frac{1-s^2}{2} f\}$$

that appeared in the study of a compact hyperbolic curve X.

Classical modular forms

(I) We will only give a sketch of proof. The key point is that the space of vectors $v \in DS_m^-$ which are of K-type m (in particular smooth) and killed by X_- is one-dimensional. Pick a generator v. Thus whenever $\varphi : DS_m^- \to H_{\mathrm{cusp}}$ is a G-equivariant map, $f = \varphi(v)$ is an element of $A_{\mathrm{cusp,m}}^{X_-=0}$, which is isomorphic to $S_m(\Gamma)$ by the previous theorem. Thus we get a map

$$\mathrm{Hom}_G(DS_m^-, H_{\mathrm{cusp}}) \to S_m(\Gamma),$$

which is injective by irreducibility of DS_m^-.

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which is injective by irreducibility of DS_m^-.

(II) Surjectivity lies deeper: one needs to check that if $f \in S_m(\Gamma)$ then the sub-representation of H_{cusp} generated by F_f is isomorphic to DS_m^-. This involves a fine study of $(\mathfrak{g}, K)$-modules, which is skipped.

Eisenstein series

(I) The orthogonal $H_{\mathrm{cusp}}^{\perp}$ of H_{cusp} in H is controlled by Eisenstein series, but in a very complicated way. For simplicity we will assume that there is a unique cusp (up to the action of Γ), situated at ∞. The associated parabolic subgroup is $P = B$ and we let as usual $\Gamma_N = \Gamma \cap N$ and $\Gamma_P = \Gamma \cap P$. We saw in a previous lecture that $\Gamma_P \subset \pm\Gamma_N$. One example to keep in mind is $\Gamma = \mathbb{SL}_2(\mathbb{Z})$.

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(II) Let

$$U = \{s \in \mathbb{C} | \operatorname{Re}(s) > 1\}.$$

Fix an integer m **and suppose that** m **is even or** $\Gamma_P = \Gamma_N$ (if not all Eisenstein series will be 0 and we won't say anything smart. Everything below depends on the choice of m, but we don't make this explicit).

Eisenstein series

(I) We will construct a map

$$E : U \to C^\infty(\Gamma\backslash G), s \to E(s),$$

where

$$E(s)(g) = \sum_{\gamma\in\Gamma_P\backslash\Gamma} \varphi_s(\gamma g),$$

for a suitable function

$$\varphi_s \in C^\infty(\Gamma_P NA\backslash G)_m.$$

More precisely,

$$\varphi(n_x a_y k) := y^{\frac{1+s}{2}} \chi_m(k).$$

More conceptually, we can write $\varphi_s = \varphi_\bullet h_P^{1+s}$, where

$$\varphi(nak) = \chi_m(k),\ h_P(nak) = \alpha_P(a)^{1/2} = \mathrm{Im}(nak.i)^{1/2},$$

where $\alpha_P : A \to \mathbb{R}_{>0}$ is the character through which A acts on $\mathrm{Lie}(N)$.

Eisenstein series

(I) A direct but tedious computation shows that

$$\mathscr{C}\varphi_s = \frac{s^2 - 1}{2}\varphi_s.$$

Theorem If $s \in U$, the series defining $E(s)(g)$ converges locally uniformly and the resulting function $E(s) \in A(\Gamma)^s_m$. The map $E : U \to C^\infty(\Gamma \backslash G)$ is holomorphic.

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(II) The proof is tricky for general lattices, the hardest point being the convergence, but this is an elementary exercise for $\mathbb{SL}_2(\mathbb{Z})$, as we will see. The fact that $E(s)$ is killed by $\mathscr{C} - \frac{s^2-1}{2}$ is an elementary consequence of the similar statement for φ_s. The moderate growth condition follows from the bounds used to prove convergence, and similarly for the holomorphic behaviour of E.

Eisenstein series

(I) Let's get a bit down to earth and study the case $m = 0$ and $\Gamma = \mathbb{SL}_2(\mathbb{Z})$. In this case $E(s)$ is right K-invariant and thus descends to a function on $\mathscr{H}$, and we will simply write $E(s, z) = E(s, g)$ if $z = g.i$. Thus

$$E(s,z) = \sum_{\gamma \in \Gamma_P \backslash \Gamma} \mathrm{Im}(\gamma.z)^{\frac{s+1}{2}} = \frac{1}{2} \sum_{(c,d) \in \mathbb{Z}^2, \gcd(c,d)=1} \frac{y^{\frac{s+1}{2}}}{|cz+d|^{s+1}}.$$

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(II) The second equality follows from the description of $\Gamma_N \backslash \Gamma$, which is identified with the set of pairs of relatively prime integers (by sending $\Gamma_N \gamma$ to the second row of γ) and the fact that $\Gamma_P = \pm \Gamma_N$.

Eisenstein series

(I) It is then very easy to check the absolute and locally uniform convergence of the series for $s \in U$ and to see that $E(s)$ has moderate growth.

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(II) Let

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}, \ \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

and

$$\Lambda(s) = \pi^{-s} \Gamma(s) \zeta(2s).$$

Write $E(s, z) = E_1(\frac{s+1}{2}, z)$, thus

$$E_1(t, z) = \frac{1}{2} \sum_{\gcd(c,d)=1} \frac{y^t}{|cz+d|^{2t}}.$$

Finally set

$$E_1^*(t, z) = \Lambda(t) E_1(t, z).$$

Eisenstein series

(I) We will prove the following:

Theorem The map $U \to C^\infty(\Gamma\backslash\mathscr{H})$ given by $t \to E_1^*(t,\bullet)$ extends to a meromorphic function on $\mathbb{C}$, holomorphic everywhere except at $0, 1$, where it has simple poles. Moreover, we have the functional equation

$$E_1^*(t,z) = E_1^*(1-t,z).$$

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$$E_1^*(t, z) = E_1^*(1 - t, z).$$

(II) It is a standard fact that Λ has meromorphic continuation to $\mathbb{C}$ with a functional equation $\Lambda(t) = \Lambda(\frac{1}{2} - t)$, thus the theorem implies that $s \to E(s, \bullet)$ also has meromorphic continuation and a functional equation. Actually the proof for E_1 (given below) is an adaptation of the proof for Λ. Unfortunately it does not adapt to general Γ, and the proof in general is much deeper.

Eisenstein series

(I) We now move to the proof. First, observe that

$$\zeta(2t)E_1^*(t,z) = \frac{1}{2}\sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}} \frac{y^t}{|cz+d|^{2t}}.$$

Eisenstein series

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(II) It follows that (there are no convergence, permutation of sums and integral issues for $t \in U$)

$$\Lambda(t)E_1^*(t,z) = \sum_{c,d} \left(\frac{y}{\pi|cz+d|^2}\right)^t \int_0^\infty e^{-u}u^t \frac{du}{u}$$

$$= \sum_{c,d} \int_0^\infty \left(\frac{uy}{\pi|cz+d|^2}\right)^t e^{-u}\frac{du}{u}$$

$$= \sum_{c,d} \int_0^\infty e^{-\pi|cz+d|^2 v/y} v^t \frac{dv}{v} = \int_0^\infty (\theta_z(v)-1)v^t \frac{dv}{v},$$

$$\theta_z(v) := \sum_{c,d\in\mathbb{Z}} e^{-\pi|cz+d|^2 v/y}.$$

Eisenstein series

(I) The Poisson summation formula applied to the function

$$A(u_1, u_2) = e^{-\pi|u_1 z + u_2|^2 t/y},$$

yields the crucial functional equation

$$\theta_z(v) = \frac{1}{v}\theta_z(v^{-1}).$$

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(II) Split the integral $\int_0^\infty (\theta_z(v) - 1)v^t \frac{dv}{v}$ in two pieces: one from 0 to 1 and the other from 1 to ∞. In the first integral make the change of variable $v \to 1/v$ and use the functional equation above. We obtain

$$\Lambda(t)E_1(t,z) = \frac{1}{2}\int_1^\infty (\theta_z(v)-1)(v^s + v^{1-s})\frac{dv}{v} + \frac{1}{2s-2} - \frac{1}{2s}.$$

We conclude by observing that since $\theta_z - 1$ has exponential decay at ∞, the integral converges for any value of $s \in \mathbb{C}$ and defines a holomorphic function of s.