Lecture 3: $\mathbb{SL}_2(\mathbb{R})$, part 1

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Goal

(I) In the next 3 or 4 lectures we will come back to earth and study more carefully the group $G = \mathbb{SL}_2(\mathbb{R})$, the automorphic forms on it and the spectral decomposition of $L^2(\Gamma\backslash G)$, where Γ is a lattice in G, as well as the link with representation theory.

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(II) The main reference will be Borel's book "Automorphic forms on $\mathbb{SL}_2$", to be called the Bible from now on. For discrete subgroups of G an excellent reference is Katok's book "Fuchsian groups". We will take for granted the geometric properties of lattices in G, which are not easy to establish in complete generality, but which are elementary for co-compact lattices and finite index subgroups of $\mathbb{SL}_2(\mathbb{Z})$, which are the main interesting examples for the automorphic theory.

Goal

(I) Here are some sources of lattices in G:

• arithmetic origin: finite index subgroups of $\mathbb{SL}_2(\mathbb{Z})$ are lattices in G, not co-compact. Also quaternion division algebras over $\mathbb{Q}$, split over $\mathbb{R}$, give rise naturally to co-compact lattices in G (we will see this in a later lecture).

• geometric origin: if X is a compact hyperbolic surface of genus ≥ 2, by the uniformization theorem there is a co-compact lattice $\Gamma \subset G$ such that $X \simeq \Gamma \backslash \mathscr{H}$, where $\mathscr{H}$ is the upper half-plane.

Structure of G

(I) There are three crucial subgroups in G:

$$A = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0\}, \ N = \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix}$$

and the standard maximal compact subgroup $K = \mathbb{SO}(2)$.

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(II) The product map $N \times A \times K \to G$ is a diffeomorphism (**Iwasawa decomposition**), in particular $\mathscr{H} \simeq G/K$ is diffeomorphic to $N \times A$. Concretely

$$z = x + iy \in \mathscr{H} \mapsto (\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}) \in N \times A.$$

Structure of G

(I) We will have to study a lot the growth of functions on G, and for this we will use the norm (for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$)

$$||g|| = \sqrt{\mathrm{tr}(gg^t)} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Note that $||g|| \geq 1$, $||gh|| \leq ||g|| \bullet ||h||$ and $||k_1 g k_2|| = ||g||$ if $k_i \in K$.

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(II) A function $f : G \to \mathbb{C}$ is said to have **moderate growth** (or simply MG) if there are constants c, N such that $|f(g)| \leq c||g||^N$ for all $g \in G$.

Calculus on G

(I) The Lie algebra $\mathfrak{g}$ of G is the space of 2×2 real matrices with trace 0. The standard basis of $\mathfrak{g}$ is given by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

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(III) The sub-algebra of $\mathrm{End}_{\mathbb{C}}(C^\infty(G))$ generated by these differential operators (when X runs through $\mathfrak{g}$) is denoted $U(\mathfrak{g})$ and called the **enveloping algebra**.

Calculus on G

(I) In $U(\mathfrak{g})$ we have the relations

$$ef - fe = h, he - eh = 2e, hf - fh = -2f$$

and $(e^n f^m h^k)_{n,m,k \geq 0}$ form a $\mathbb{C}$-basis of $U(\mathfrak{g})$ (Poincaré-Birkhoff-Witt theorem).

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(II) The center of $U(\mathfrak{g})$ is $\mathbb{C}[\mathscr{C}]$ where the **Casimir operator** is

$$\mathscr{C} = \frac{1}{2}h^2 + ef + fe.$$

(III) The following easy result is very useful:

$$D.(f * \alpha) = f * (D.\alpha), \ \forall f \in C^\infty(G), \alpha \in C_c^\infty(G), \ D \in U(\mathfrak{g}).$$

Indeed, this reduces to the case $D \in \mathfrak{g}$, and then

$$D(f * \alpha)(x) = \frac{d}{dt}|_{t=0} \int_G f(xe^{tD}y^{-1})\alpha(y)dy$$

$$= \frac{d}{dt}|_{t=0} \int_G f(xz^{-1})\alpha(ze^{tD})dz = f * (D.\alpha)(x).$$

Calculus on G

(I) Say $f \in L^1_{\text{loc}}(G)$, i.e. f is locally integrable on G. If $D \in \mathbb{C}[\mathscr{C}]$ we write Df for the distribution

$$(Df)(\varphi) = \int_G f(x)(D\varphi)(x)dx, \;\; \varphi \in C_c^\infty(G).$$

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(II) We say that f is $\mathscr{C}$-**finite** if there is $P \in \mathbb{C}[X]$ nonconstant such that $P(\mathscr{C})f = 0$. If f is smooth, there is no need to talk about distributions.

Two deep results

(I) We will use the following two hard results, the first one being an easy consequence of a hard analytic theorem called elliptic regularity. The second one will be proved in much greater generality later on.

Theorem Let $f \in L^1_{\mathrm{loc}}(G)$ a $\mathscr{C}$-finite and right K-finite function. Then:

- (elliptic regularity) f is real analytic in G.

- (Harish-Chandra's harmonicity theorem) there is $\alpha \in C_c^\infty(G)$ such that $f = f * \alpha$.

We can take α invariant by conjugation by K, with support contained in a given neighborhood of 1 in G.

Automorphic forms on G

(I) The space $A(\Gamma)$ of **automorphic forms of level** Γ (for G) is the space of functions $f \in C^{\infty}(\Gamma \backslash G)$ which are right K-finite, $\mathscr{C}$-finite and of moderate growth. The MG condition is automatic if $\Gamma \backslash G$ is compact, and in general it is imposed to avoid explosion at "cusps" of $\Gamma \backslash \mathscr{H}$ (this notion will be discussed a bit later on this lecture).

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(II) As a special case of the hard theorems mentioned above:

Theorem Any $f \in A(\Gamma)$ is real analytic and there is $\alpha \in C_c^\infty(G)$ such that $f = f * \alpha$.

Automorphic forms on G

(I) It is immediate that $A(\Gamma)$ is stable under the right translation action of K. It is not stable under G (the right K-finiteness is lost), but the following consequence of the previous theorem shows that it is stable under $\mathfrak{g}$ (and forms a $(\mathfrak{g}, K)$-module, animals that will be studied a lot in later lectures):

Theorem

a) If $f \in A(\Gamma)$, there is N such that for all $D \in U(\mathfrak{g})$ we have

$$\sup_{g \in G} \frac{|D.f(g)|}{||g||^N} < \infty.$$

b) If $f \in A(\Gamma)$, then $D.f \in A(\Gamma)$ for all $D \in U(\mathfrak{g})$.

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b) If $f \in A(\Gamma)$, then $D.f \in A(\Gamma)$ for all $D \in U(\mathfrak{g})$.

(II) Part b) immediately follows from a): Df has moderate growth by a) and the other properties are easy.

Automorphic forms on G

(I) Write $f = f * \alpha$ for some $\alpha \in C_c^\infty(G)$ (previous theorem!). Pick c, N such that $|f(g)| \leq c||g||^N$ for all g. We have

$$||Df(g)|| = ||D(f * \alpha)(g)| = |f * (D.\alpha)(g)| \leq$$

$$\int_G c||gx^{-1}||^N |(D.\alpha)(x)| dx \leq c||g||^N \int_G ||x^{-1}||^N |(D.\alpha)(x)| dx,$$

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(II) Since $D.\alpha \in C_c^\infty(G)$, the last integral is finite and so we are done.

Examples of automorphic forms

(I) The easiest way to produce automorphic forms is via Poincaré series. However, proving that the resulting functions really are automorphic forms is not that easy (it uses the harmonicity theorem):

Theorem Let $\varphi \in L^1(G)$ be a (right) K-finite and $\mathscr{C}$-finite function and consider the map $p_\varphi : G \to \mathbb{C}$

$$p_\varphi(x) = \sum_{\gamma \in \Gamma} \varphi(\gamma x).$$

The series converges absolutely and locally uniformly and $p_\varphi \in A(\Gamma) \cap L^1(\Gamma \backslash G)$.

Examples of automorphic forms

(I) The proof is rather indirect (the one below is slightly different than the one in the Bible). We start with the following technical result:

Lemma Given $\alpha \in C_c^\infty(G)$ there are $c, N > 0$ such that for all $\varphi \in L^1(G)$ and all $x \in G$ we have

$$\sum_{\gamma \in \Gamma} |(\varphi * \alpha)(\gamma x)| \leq c||x||^N ||\varphi||_{L^1(G)}.$$

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(II) Let's see why this implies the theorem. By the harmonicity theorem there is $\alpha \in C_c^\infty(G)$ such that $\varphi = \varphi * \alpha$. For any $D \in U(\mathfrak{g})$, applying the lemma to $D.\alpha \in C_c^\infty(G)$ and using the relation $D\varphi = \varphi * (D.\alpha)$, we obtain the absolute and locally uniform convergence of $\sum_\gamma (D.\varphi)(\gamma x)$. Thus p_φ is well-defined, smooth and $(D.p_\varphi)(x) = \sum_\gamma (D.\varphi)(\gamma x)$. Since φ is $\mathscr{C}$-finite, so is p_φ. Similarly for right K-finiteness.

Examples of automorphic forms

(I) Left Γ-invariance is clear, and moderate growth follows from the lemma. Finally $p_\varphi \in L^1(\Gamma\backslash G)$ since

$$\int_{\Gamma\backslash G} |p_\varphi(x)| dx \leq \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} |\varphi(\gamma x)| dx = \int_G |\varphi(x)| dx < \infty.$$

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(II) Let's prove the lemma now. Let U a compact set containing $\mathrm{Supp}(\alpha)$, then

$$|(\varphi * \alpha)(\gamma x)| \leq \int_G |\varphi(z)||\alpha(z^{-1}\gamma x)|dz$$

$$\leq ||\alpha||_\infty \int_G 1_{z^{-1}\gamma x\in U}|\varphi(z)|dz.$$

Examples of automorphic forms

(I) If we can prove that for suitable c, N we have

$$\sum_{\gamma \in \Gamma} 1_{z^{-1}\gamma x \in U} \leq c||x||^N$$

for all x, z, then we are done: by the previous inequalities

$$\sum_{\gamma \in \Gamma} |\varphi(\gamma x)| \leq c||x||^N \int_G |\varphi(z)| dz.$$

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(II) If $z^{-1}\gamma_i x \in U$ for $1 \leq i \leq d$, then $x^{-1}\gamma_i^{-1}\gamma_1 x \in U^{-1}U$ for $1 \leq i \leq d$. Since U is bounded and $||x^{-1}|| = ||x||$, we obtain $||\gamma_i^{-1}\gamma_1|| \leq c||x||^2$ for a constant c depending only on U.

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(III) We finish the proof using the following nice

Lemma There are constants c, N such that for all $r > 0$ there are at most cr^N elements $\gamma \in \Gamma$ with $||\gamma|| \leq r$.

Examples of automorphic forms

(I) This is very easy if $\Gamma \subset \mathbb{SL}_2(\mathbb{Z})$, as then the entries of γ take at most $2r + 1 \leq 3r$ (if $r \geq 1$, which we may assume) different values (they are between $-r$ and r and are integers), so we can take $c = 3^4$ and $N = 4$ in this case.

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(II) In general, since Γ is discrete, there is a relatively compact open neighborhood U of 1 such that $UU^{-1} \cap \Gamma = \{1\}$. Let $B_r = \{x \in G| \, ||x|| \leq r\}$. If $\gamma_i \in \Gamma \cap B_r$ for $1 \leq i \leq d$, then $\gamma_i U$ are pairwise disjoint and contained in B_{rc} with $c = \max_{u \in \bar{U}} ||u||$. Thus $d\mathrm{vol}(U) \leq \mathrm{vol}(B_{rc})$ and it suffices to show that $r \to \mathrm{vol}(B_r)$ grows at most polynomially. This is not hard, cf Bible lemma 5.12.

Cuspidality

(I) **From now on we assume that Γ is a lattice in** G. The most well-behaved analytically (and the most mysterious...) automorphic forms are the cuspidal ones. The notion of cuspidality is related to the non compactness of $\Gamma\backslash G$, or equivalently of $\Gamma\backslash\mathscr{H}$ and to the presence of nontrivial unipotent matrices in Γ.

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(II) The action of G on $\mathscr{H}$ extends to

$$\overline{\mathscr{H}} = \mathscr{H} \cup \mathbb{R} \cup \{\infty\}$$

and preserves the **boundary**

$$\partial\mathscr{H} = \mathbb{R} \cup \{\infty\}$$

of $\mathscr{H}$, on which $K/\{\pm 1\}$ acts simply transitively.

Cuspidality

(I) The stabiliser of a point of $\partial\mathscr{H}$ is called a **parabolic subgroup** of G. The **standard parabolic** is the stabiliser of ∞, namely

$$B = \pm NA = \{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}^*, b \in \mathbb{R} \}.$$

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(II) For any parabolic P there is $k \in K$ (unique up to ± 1) with $kBk^{-1} = P$. Let $A_P = kAk^{-1}$ (thus $A_B = A$). The **unipotent radical** $N_P = kN_B k^{-1}$ **of** P and A_P are independent of the choice of k, N_P is normal in P and $N_P \times A_P \times K \to G$ is a diffeomorphism.

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(III) Let $z \in \partial\mathscr{H}$ and let $P = G_z$ be its stabiliser and $N = N_P$ the unipotent radical of P (i.e. the unipotent matrices in P). We say that z is a **Γ-cuspidal point** (and that P is a **Γ-cuspidal parabolic**) if $\Gamma \cap N \neq \{1\}$.

Cuspidality

(I) We let $C(\Gamma)$ (resp. $CP(\Gamma)$) be the set of Γ-cuspidal points (resp. parabolic subgroups). Γ acts naturally on $C(\Gamma)$ and $CP(\Gamma)$. Thus a point $z \in \partial\mathscr{H}$ is in $C(\Gamma)$ if and only if z is fixed by some nontrivial unipotent element of Γ.

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(II) If $P \in CP(\Gamma)$, then $\Gamma \cap N$ is an infinite cyclic group and $\Gamma \cap P \subset \pm(\Gamma \cap N)$.

Cuspidality

(I) Indeed, by conjugating WLOG $z = \infty$ so $P = B$. Then $\Gamma \cap N$ is identified with a nontrivial discrete subgroup of $\mathbb{R}$, thus $\Gamma \cap N = \begin{pmatrix} 1 & h\mathbb{Z} \\ 0 & 1 \end{pmatrix}$ for some $h > 0$. If $\gamma = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \Gamma \cap P$, then conjugation by γ is a permutation of $\Gamma \cap N$ and given by $\gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \gamma^{-1} = \begin{pmatrix} 1 & a^2 x \\ 0 & 1 \end{pmatrix}$, thus $a^2 = 1$.

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(II) In particular, if $\Gamma' \subset \Gamma$ has finite index, then $CP(\Gamma) = CP(\Gamma')$ and $C(\Gamma) = C(\Gamma')$ (use that $\Gamma \cap N / \Gamma' \cap N$ injects into Γ/Γ', so it is finite). It is easy to see that

$$C(\mathbb{SL}_2(\mathbb{Z})) = \mathbb{Q} \cup \{\infty\}$$

and thus the same holds for any finite index subgroup of $\mathbb{SL}_2(\mathbb{Z})$.

Cuspidality

(I) Let

$$\mathscr{H}_\Gamma^* = \mathscr{H} \cup C(\Gamma).$$

There is a natural topology on this space, for which $\mathscr{H}$ is an open subspace and a fundamental system of neighborhoods of $z \in C(\Gamma)$ consists of the closed discs contained in $\overline{\mathscr{H}}$ and tangent to $\partial\mathscr{H}$ at z (if $z = \infty$ this is to be interpreted as $\{\infty\} \cup \{z \in \mathscr{H} \,|\, \mathrm{Im}(z) > t\}$, for some $t > 0$).

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(II) See prop 3.10 in the Bible for the nontrivial proof of:

Theorem For any discrete subgroup Γ of G, the quotient space

$$X(\Gamma) = \Gamma \backslash \mathscr{H}_\Gamma^*$$

is locally compact, thus Hausdorff.

Cuspidality

(I) The previous theorem is related to a very classical result of Poincaré, saying that any discrete subgroup Γ acts properly on $\mathscr{H}$, i.e. for any compact subset $C \subset \mathscr{H}$ the set $\{\gamma \in \Gamma | \gamma C \cap C \neq \emptyset\}$ is finite, thus by general nonsense the topological space

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(II) When Γ is a lattice in G, the next deep theorem shows that $X(\Gamma)$ gives a compactification of $Y(\Gamma)$, by adding finitely many points to it, called the cusps of $X(\Gamma)$ (they are in bijection with $\Gamma \backslash C(\Gamma)$).

Cuspidality

(I) See the Bible 3.13, 3.14 for the rather delicate proof of the next theorem. For Γ a finite index subgroup of $\mathbb{SL}_2(\mathbb{Z})$ the proof is much easier and left as an excellent exercise.

Theorem (Siegel) For any lattice Γ in G we have:
a) The sets $\Gamma\backslash C(\Gamma)$ and $\Gamma\backslash CP(\Gamma)$ are finite.
b) $X(\Gamma)$ is compact.
c) $Y(\Gamma)$ is compact if and only if Γ is co-compact in G, if and only if $C(\Gamma) = \emptyset$.
d) Γ is finitely generated.

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(II) Let $P \in CP(\Gamma)$ and $N = N_P$ its unipotent radical. For any $f \in L^1_{\mathrm{loc}}(\Gamma \cap N \backslash G)$ we define its **constant term along** P as

$$f_P(g) = \int_{\Gamma \cap N \backslash N} f(ng)dn,$$

where dn is normalised so that $\mathrm{vol}(\Gamma \cap N \backslash N) = 1$. This is well-defined for almost all g, and locally integrable, by Fubini's theorem.

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(III) We say that f is **cuspidal at** P (or at the point of $\partial \mathscr{H}$ fixed by P) if f_P is the zero map.

Cuspidal automorphic forms

(I) The space of **cuspidal automorphic forms of level Γ** is

$$A_{\mathrm{cusp}}(\Gamma) = \{f \in A(\Gamma) |\, f_P = 0,\ \forall P \in CP(\Gamma).\}$$

To check that $f \in A(\Gamma)$ is cuspidal, it suffices (exercise) to check that $f_P = 0$ for a set of representatives of $\Gamma \backslash CP(\Gamma)$, which is finite by Siegel's theorem.

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(II) We can also define the cuspidal subspace of $L^2(\Gamma \backslash G)$

$$L^2_{\mathrm{cusp}}(\Gamma \backslash G) = \{f \in L^2(\Gamma \backslash G) | \, f_P(g) = 0 \, a.e.g, \forall P \in \mathrm{CP}(\Gamma)\}.$$

Cuspidal automorphic forms

(I) The proof of the following result is fairly delicate, cf. Bible th 8.9:

Theorem Let $\varphi \in L^1(G)$ be a $\mathscr{C}$-finite, left and right K-finite function. Then the Poincaré series $p_\varphi \in A_{\mathrm{cusp}}(\Gamma)$.

Classical modular forms

(I) We take a break from automorphic forms and introduce classical modular forms. These will turn out to yield other very interesting examples of automorphic forms.

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(II) Let k be an integer. Define

$$\mu(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) = cz + d,$$

then $\mu(gh, z) = \mu(g, hz)\mu(h, z)$ for all g, h, z, thus setting

$$(f|_k g)(z) = f(gz)\mu(g, z)^{-k}$$

defines a right action of G on the space $\mathscr{O}(\mathscr{H})$ of holomorphic functions on $\mathscr{H}$.

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defines a right action of G on the space $\mathscr{O}(\mathscr{H})$ of holomorphic functions on $\mathscr{H}$.

(III) The space $WM_k(\Gamma)$ of **weakly-modular forms of level** Γ **and weight** k consists of those $f \in \mathscr{O}(\mathscr{H})$ that are Γ-invariant under the above action, i.e.

$$f(\gamma.z) = \mu(\gamma, z)^k f(z), \quad \gamma \in \Gamma, z \in \mathscr{H}.$$

Classical modular forms

(I) For instance, if $\Gamma = \mathbb{SL}_2(\mathbb{Z})$ we have $f \in WM_k(\Gamma)$ if and only if $f(z+1) = f(z)$ and $f(-1/z) = z^k f(z)$, since $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $\mathbb{SL}_2(\mathbb{Z})$.

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(II) Take f a weakly modular form of weight k and level Γ and suppose that $\infty \in C(\Gamma)$, thus $\Gamma \cap \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & h\mathbb{Z} \\ 0 & 1 \end{pmatrix}$ for some $h > 0$. Then $f(z+h) = f(z)$.

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(III) Since $\begin{pmatrix} 1 & h\mathbb{Z} \\ 0 & 1 \end{pmatrix} \backslash \mathscr{H}$ is biholomorphic to $D^* = \{z \in \mathbb{C} | \, 0 < |z| < 1\}$ via $z \to e^{2i\pi z/h}$, there are $a_n \in \mathbb{C}$ and an absolutely and locally uniform convergent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n q_h^n, \ , \ q_h = e^{2i\pi z/h},$$

called the q-**expansion of** f **at infinity**.

Classical modular forms

(I) We say that f is **holomorphic at ∞** (resp. **vanishes at ∞**) if $a_n = 0$ for $n < 0$ (resp. for $n \leq 0$).

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(II) Now let $c \in C(\Gamma)$ be arbitrary and let $g \in G$ be such that $g.\infty = c$. Now $f|_k g \in WM_k(g^{-1}\Gamma g)$ and $\infty \in C(g^{-1}\Gamma g)$, so we can give a meaning to f being holomorphic (resp. vanishing) at c, by asking that this should happen for $f|_k g$ at ∞. This is well-defined, i.e. independent of the choice of g such that $g.\infty = c$ (excellent exercise in bookkeeping), even though the q-expansion at ∞ of $f|_k g$ depends on g.

Classical modular forms

(I) We define then the space $M_k(\Gamma)$ of **modular forms of level Γ and weight k** as the space of weakly modular forms of level Γ and weight k, which are holomorphic at all cuspidal points of Γ. Similarly define the space $S_k(\Gamma)$ of **cuspidal modular forms of level Γ and weight k**. We will see in the next lecture that it naturally embeds in $A(\Gamma)$.

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(II) Let's give some classical examples of modular forms. We will take $\Gamma = \mathbb{SL}_2(\mathbb{Z})$, for simplicity. If $k \geq 3$ simple arguments show that for any 1-periodic bounded $\varphi \in \mathscr{O}(\mathscr{H})$ the modified Poincaré series

$$P_{k,\varphi}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi(\gamma z) \in M_k(\Gamma).$$

(I) For $\varphi = 1$ we write $E_k = P_{k,\varphi}$ the **normalised Eisenstein series of weight** k

$$E_k(z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{1}{(cz+d)^k} = \frac{1}{2\zeta(k)} G_k(z),$$

with G_k the classical Eisenstein series of weight k

$$G_k(z) = \sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(cz+d)^k}.$$

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(II) **Euler's identity**, valid for $k \geq 2$ with $q = e^{2i\pi z}$

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+z)^k} = \frac{(-2i\pi)^k}{(k-1)!} \sum_{d\geq 1} d^{k-1} q^d$$

is obtained by differentiating $k-1$ times the classical Euler identity

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n\geq 1} \left(\frac{1}{z-n} + \frac{1}{z+n} \right).$$

(I) This immediately yields the q-expansion of G_k at ∞:

$$E_k(z) = 1 + \frac{(-2i\pi)^k}{\zeta(k)} \frac{1}{(k-1)!} \sum_{n\geq 1} \sigma_{k-1}(n) q^n,$$

where $\sigma_s(n) = \sum_{d|n, d>0} d^s$ and $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$.

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(II) For instance

$$E_4 = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n, \ \ E_6(z) = 1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n.$$

We will see later on that any modular form of any weight for $\mathbb{SL}_2(\mathbb{Z})$ is a polynomial in E_4 and E_6 (and these are algebraically independent).

(I) The q-expansion of E_k has rational coefficients thanks to Euler's classical result

$$\frac{\zeta(k)}{(2i\pi)^k} \in \mathbb{Q}, \;\; k \in \{2, 4, 6, \ldots\},$$

deduced by rewriting his identity as

$$1 - i\pi z - \frac{2i\pi z}{e^{2i\pi z} - 1} = 1 - \pi \cot(\pi z) = 2z^2 \sum_{n \geq 1} \frac{1}{n^2 - z^2}$$

$$= 2z^2\zeta(2) + 2z^4\zeta(4) + 2z^6\zeta(6) + \ldots$$

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$$= 2z^2\zeta(2) + 2z^4\zeta(4) + 2z^6\zeta(6) + \dots$$

(II) The double series $\sum_{c,d} \frac{1}{(cz+d)^2}$ does not converges absolutely, but (exclude $(c,d) = (0,0)$ in the sum below)

$$G_2(z) := \sum_{c \in \mathbb{Z}} \left(\sum_{d \in \mathbb{Z}} \frac{1}{(cz+d)^2} \right)$$

converges and Euler's identity still gives

$$G_2(z) = \frac{\pi^2}{3}(1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n).$$

(I) However, G_2 is NOT a modular form. A rather subtle algebraic manipulation shows that

$$G_2(-1/z) = z^2 G_2(z) - 2i\pi z.$$

This implies that $z \to G_2(z) - \frac{\pi}{\mathrm{Im}(z)}$ is $\mathbb{SL}_2(\mathbb{Z})$-invariant for the $|_2$-action, BUT... it is not holomorphic!

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(II) Still, the relation above has the following amazing consequence:

Theorem (Jacobi) The following function Δ gives an element of $S_{12}(\mathbb{SL}_2(\mathbb{Z}))$, where $q = e^{2i\pi z}$

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24}.$$

(I) The only tricky part is showing that $\Delta(-1/z) = z^{12}\Delta(z)$ (and this is really damn tricky!). A simple calculation shows that

$$\frac{\Delta'(z)}{\Delta(z)} = 2i\pi(1 - 24\sum_{n\geq 1}\frac{q^n}{1-q^n}) = 2i\pi(1 - 24\sum_{n\geq 1}\sigma_1(n)q^n),$$

thus up to a constant this is $G_2(z)$.

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(II) The relation between $G_2(-1/z)$ and $G_2(z)$ immediately yields $f'(z)/f(z) = 0$, where $f(z) = \frac{\Delta(-1/z)}{z^{12}\Delta(z)}$. Thus f is constant and since $f(i) = 1$, we have $f = 1$ and we are done!