Lecture 2: operators on Hilbert spaces and applications

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Abstract nonsense

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(II) The space $C_c(G)$ has a ring structure via the **convolution product**

$$f_1 * f_2(x) = \int_G f_1(xg^{-1}) f_2(g) dg.$$

Theorem Any $V \in \mathrm{Rep}(G)$ has a natural structure of $C_c(G)$-module, denoted $(f, v) \to f.v = \int_G f(g) g.v dg$, such that for all $f \in C_c(G)$ and any continuous linear form l on V we have

$$l(f.v) = \int_G f(g) l(g.v) dg.$$

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(I) I will only discuss the case of Hilbert representations V. Given $f \in C_c(G)$, the map sending $l \in V^*$ (topological dual) to $\int_G f(g) l(g.v) dg$ is a continuous linear form on V^*, so by Riesz' theorem there is a unique $f.v \in V$ such that $l(f.v) = \int_G f(g) l(g.v) dg$ for all $l \in V^*$. One easily checks that $(f_1 * f_2).v = f_1.(f_2.v)$ and $(f_1 + f_2).v = f_1.v + f_2.v$ (test these against arbitrary $l \in V^*$).

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(II) A **Dirac sequence on** G is a sequence of functions $f_n \in C_c(G)$ such that for all j we have:
• $f_j(g) \geq 0$, $f_j(g^{-1}) = f_j(g)$ for all g and $\int_G f_j(g)dg = 1$.
• $\mathrm{Supp}(f_j)$ form a decreasing sequence "tending to $\{1\}$" in an obvious sense.

Abstract nonsense

(I) Dirac sequences always exist, and given a compact subgroup K of G we can choose them such that $f_n(kgk^{-1}) = f_n(g)$ for $k \in K$ and $g \in G$. If G is a Lie group, we can pick f_n smooth as well.

Theorem If $V \in \mathrm{Rep}(G)$, $v \in V$ and (f_n) is a Dirac sequence, then $\lim_{n\to\infty} f_n.v = v$.

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Theorem If $V \in \mathrm{Rep}(G)$, $v \in V$ and (f_n) is a Dirac sequence, then $\lim_{n\to\infty} f_n.v = v$.

(II) Suppose that V is a Hilbert representation. Given $\varepsilon > 0$ there is a neighborhood U of 1 such that $||g.v - v|| \leq \varepsilon$ for $g \in U$. For n large enough we have $\mathrm{Supp}(f_n) \subset U$ and

$$||f_n.v - v|| = ||\int_G f_n(g)(g.v - v)dg|| \leq \int_G f_n(g)||g.v - v||dg$$

$$\leq \varepsilon \int_G f_n = \varepsilon.$$

Operators on Hilbert spaces

(I) Let H be a separable complex Hilbert space. An **operator** on H is a continuous linear map $T : H \to H$. Any operator T has an **adjoint operator** T^*, characterised by $\langle Tv, w\rangle = \langle v, T^*w\rangle$ for $v, w \in H$ (apply Riesz!).

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(II) For instance, if H is a unitary representation of some G, and if $f \in C_c(G)$, the adjoint of the operator $T_f : H \to H, v \to f.v = \int_G f(g)g.vdg$ is T_{f^*}, where $f^*(g) = \overline{f(g^{-1})}$ (easy computation).

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(III) The space $B(H)$ of operators on H is a Banach algebra for the norm $||T|| = \sup_{v\neq 0} ||Tv||/||v||$. The operator $T \in B(H)$ is called **self-adjoint** if $T = T^*$, **unitary** if $TT^* = T^*T = \mathrm{id}$ (i.e. T is an isometry), **positive** if $\langle Tv, v\rangle \geq 0$ for all v (such a T is then self-adjoint) and finally **normal** if T commutes with T^*.

Operators on Hilbert spaces

(I) The **spectrum** of $T \in B(H)$ is

$$\sigma(T) = \{\lambda \in \mathbb{C} |\, \lambda - T \text{ is not invertible}\}.$$

By Gelfand's theory $\sigma(T)$ is a compact subset of $\mathbb{C}$ and

$$\max_{\lambda \in \sigma(T)} |\lambda| = \lim_{n\to\infty} ||T^n||^{1/n},$$

the spectral radius of T. If T is normal, then $\lim_{n\to\infty} ||T^n||^{1/n} = ||T||$, since $||T^2|| = ||T||^2$.

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(II) Suppose now that T is self-adjoint and let $K = \sigma(T)$. Then K is compact in $\mathbb{R}$, so (Stone-Weierstrass) any $f \in C(K)$ is a limit of polynomial functions p_n. The operators $p_n(T)$ converge to an operator $f(T) \in B(H)$ (use that for $p \in \mathbb{C}[T]$ is normal, thus $||p(T)|| = \max_{x \in K} |p(x)|$). This yields an isometric morphism of Banach algebras $C(K) \to B(H), f \to f(T)$ (**functional calculus**).

Operators on Hilbert spaces

(I) An operator $T \in B(H)$ is called **compact** if T sends bounded subsets of H to relatively compact subsets, or equivalently T is a limit (in $B(H)$) of operators of finite rank. The set $K(H)$ of compact operators is closed in $B(H)$ and forms a two-sided ideal in $B(H)$.

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(II) Say now $\dim H = \infty$. Then for any compact operator T we have $0 \in \sigma(T)$ and $\sigma(T) \setminus \{0\}$ is at most countable and consists of eigenvalues of T. The eigenspaces corresponding to nonzero eigenvalues are finite dimensional. If T is moreover normal, then $\ker(T)^\perp$ has an ON-basis of eigenvectors, and the corresponding eigenvalues tend to 0.

Operators on Hilbert spaces

(I) An operator $T \in B(H)$ is called **Hilbert-Schmidt** (or simply HS), respectively **of trace class** (or simply TC) if H has an ON-basis $(e_n)_n$ such that $\sum_n ||Te_n||^2 < \infty$, respectively $\sum_n ||Te_n|| < \infty$.

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(II) Let $HS(H)$, resp. $TC(H)$ be the sets of HS, resp. trace class operators on H.

Theorem 1) We have $TC(H) = \{AB|\, A, B \in HS(H)\}$ and $HS(H) \subset K(H)$ (thus $TC(H) \subset K(H)$).
2) $T \in B(H)$ is in $TC(H)$ if and only if $\sum_n |\langle Te_n, f_n\rangle| < \infty$ for any ON-bases $(e_n)_n$ and $(f_n)_n$ of H, and $\mathrm{Tr}(T) := \sum_n \langle Te_n, e_n\rangle$ converges absolutely and is independent of the choice of the ON-basis.

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(II) Here is a key example of HS operators:

Theorem (Hilbert-Schmidt) If (X, μ) is a measure space such that $H = L^2(X, \mu)$ is separable and if $K \in L^2(X \times X)$ then the operator $T_K \in B(H)$ defined by

$$T_K(f)(x) = \int_X K(x, y) f(y) d\mu(y)$$

is HS.

(I) The proof is easy: pick an ON-basis (e_n) of H. By Fubini $K(x,\bullet) \in L^2(X)$ for almost all x and $T_K(e_n)(x) = \langle K(x,\bullet), \overline{e_n}\rangle$, thus (using Plancherel and Fubini, and noting that $\overline{e_n}$ also form an ON-basis)

$$\sum ||T_K(e_n)||^2 = \sum \int_X |\langle K(x,\bullet), \overline{e_n}\rangle|^2 d\mu(x)$$

$$= \int_X ||K(x,\bullet)||^2_{L^2(X)} d\mu(x) = ||K||^2_{L^2(X\times X)} < \infty.$$

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(II) As a concrete example, let G be as usual and let Γ be a closed unimodular subgroup in G **such that** $X = \Gamma \backslash G$ **is compact** (e.g. Γ is a co-compact lattice). Let $H = L^2(X)$ with the natural action of G. For $f \in C_c(G)$ let T_f be the operator $\varphi \to f.\varphi = (x \to \int_G f(g)\varphi(xg)dg)$.

Operators on Hilbert spaces

(I) We compute

$$T_f(\varphi)(x) = \int_G f(x^{-1}g)\varphi(g)dg = \int_{\Gamma\backslash G} \varphi(g)\left(\int_\Gamma f(x^{-1}\gamma g)d\gamma\right)dg$$

$$= \int_X K_f(x,y)\varphi(y)dy, \quad K_f(x,y) = \int_\Gamma f(x^{-1}\gamma y)d\gamma.$$

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(II) Now $K_f \in C^0(X \times X) \subset L^2(X \times X)$ (as X is compact!), thus T_f is HS by the above theorem. The next theorem is MUCH deeper:

Theorem (Dixmier-Malliavin) If G is moreover a real Lie group, then $T_f \in TC(L^2(\Gamma\backslash G))$ for all $f \in C_c^\infty(G)$ and

$$\mathrm{Tr}(T_f) = \int_X K_f(x,x)dx.$$

Operators on Hilbert spaces

(I) This follows from an amazing theorem of Dixmier-Malliavin, saying that any $f \in C_c^\infty(G)$ is a finite sum of functions of the form $f_1 * f_2$ with $f_i \in C_c^\infty(G)$. Now $T_{f_1 * f_2} = T_{f_1} T_{f_2}$ and $T_{f_i} \in HS$, thus $T_{f_1 * f_2} \in TC$ and $T_f \in TC$. The computation of the trace for $f = f_1 * f_2$ is a simple computation (exercise) with ON-bases and the general case follows.

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(II) We will now move on to applications of these very general results to representation theory.

Application 1: Schur's lemma

(I) The following result is fundamental (and the proof is much subtler than for finite groups!). Keep a general G for now (so locally compact, unimodular, countable at infinity):

Theorem (Schur's lemma) For any $V \in \hat{G}$ we have $\mathrm{End}_G(V) = \mathbb{C}$, i.e. all G-equivariant endomorphisms are scalar.

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(II) Let $A = \mathrm{End}_G(V)$, a closed $\mathbb{C}$-subalgebra of $B(H)$, stable by passage to adjoints (by unitarity of V). If $T \in A$, then $T = \frac{T+T^*}{2} + i_\bullet \frac{T-T^*}{2i}$ and $\frac{T+T^*}{2}, \frac{T-T^*}{2i}$ are self-adjoint, so it suffices to prove that any self-adjoint $T \in A$ is scalar, i.e. that its spectrum $K = \sigma(T)$ has one point (as then $T - \lambda$ is self-adjoint and $\sigma(T - \lambda) = \{0\}$, thus $T = \lambda$).

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(III) If $|K| \geq 2$, one easily shows that there are $f, g \in C(K)$ nonzero such that $fg = 0$.

Schur's lemma for unitary representations

(I) Note that $f(T) \in A$ for $f \in C(K)$ (since $f(T)$ is a limit of polynomials in T and A is closed in $B(H)$). Also $f(T)g(T) = (fg)(T) = 0$. If $f(T), g(T) \neq 0$, then $\ker(f(T))$ is a sub-representation of V different from 0 and V, contradicting the irreductibility of V. So WLOG $f(T) = 0$. But then $f = 0$, since $T \to f(T)$ is an isometry, a contradiction.

Discrete decompositions

(I) Let $H_1, H_2, \ldots$ be separable Hilbert spaces. Their **Hilbert sum** $H = \widehat{\oplus}_n H_n$ is the Hilbert space obtained by completing $\oplus_n H_n$ with respect to the hermitian product

$$\langle (x_n)_n, (y_n)_n \rangle = \sum_n \langle x_n, y_n \rangle.$$

Concretely, H is the space of sequences $(x_n)_n$ with $x_n \in H_n$ and $\sum_n ||x_n||^2 < \infty$ (with the hermitian product above).

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(II) If H_n are unitary representations of some G, then so is $H = \widehat{\oplus}_n H_n$ (via $g.(x_n)_n = (g.x_n)_n$).

(III) We say that a unitary representation H of G has a **discrete decomposition** if there are irreducible unitary sub-representations H_n with $H = \widehat{\oplus}_n H_n$ and each occurs with finite multiplicity, i.e. any $\pi \in \hat{G}$ is isomorphic to only finitely many H_n.

Discrete decompositions

(I) Equivalently (use Schur's lemma) a unitary rep. H has discrete decomposition if we can write

$$H \simeq \widehat{\bigoplus}_{\pi \in \hat{G}} \pi^{\oplus m(\pi)} \simeq \widehat{\bigoplus}_{\pi \in \hat{G}} \pi \otimes \mathrm{Hom}_G(\pi, H)$$

with $m(\pi) = \dim \mathrm{Hom}_G(\pi, H) < \infty$. The following theorem is fundamental:

Theorem (Gelfand-Graev, Piatetski-Shapiro) If H is a unitary representation of G such that T_f is a compact operator on H for all $f \in C_c(G)$, then H has a discrete decomposition.

If G is a real Lie group, it suffices to impose the condition for $f \in C_c^\infty(G)$ (as the proof shows).

Discrete decompositions

(I) The main step is showing that any nonzero sub-representation W contains an irreducible sub-representation. For this, we start by picking (use Dirac sequences) $f \in C_c(G)$ such that $T := T_f|_W$ is nonzero and self-adjoint. As T is also compact, it has a nonzero eigenvalue λ. Among stable subspaces V of W for which $V[\lambda] := \ker(T - \lambda)$ is nonzero, pick one that minimises $\dim V[\lambda]$, and pick $v \in V[\lambda]$ nonzero.

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(II) We claim that $V_1 = \overline{\mathrm{Span}(G.v)}$ is irreducible. If not, $V_1 = U_1 \oplus U_2$, orthogonal sum of nonzero sub-representations. Then U_i are stable under T and $V_1[\lambda] = U_1[\lambda] \oplus U_2[\lambda]$. By minimality of V one of $U_i[\lambda]$ is 0 so WLOG $v \in U_1$, but then $V_1 \subset U_1$ and $U_2 = 0$, a contradiction.

Discrete decompositions

(I) Next we show that H is a Hilbert direct sum of irreducible sub-representations. A set of irreducible and pairwise orthogonal sub-reps. of H is called an orthogonal family. One easily checks (use Zorn's lemma) that there is a maximal orthogonal family A. The orthogonal W of $\sum_{\pi \in A} \pi$ (equivalently of $\hat{\bigoplus}_{\pi \in A} \pi$) is a sub-representation containing no irreducible sub-representation (by maximality of A), thus by the first step $W = 0$ and $H = \hat{\bigoplus}_{\pi \in A} \pi$.

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(II) Finally, we check that multiplicities are finite. Say $\pi_1, \ldots, \pi_n$ are irreducible, pairwise isomorphic, and all appear in H. Pick $f \in C_c(G)$ such that T_f is self-adjoint and nonzero on π_1, and pick a nonzero eigenvalue λ of T_f on π_1. The eigenspaces $\pi_i[\lambda]$ are all isomorphic to $\pi_1[\lambda]$ (as $\pi_i \simeq \pi_1$), in particular nonzero, and are in direct sum inside $H[\lambda]$, thus $n \leq \dim H[\lambda] < \infty$ (as T_f is compact).

Discrete decompositions

(I) Combining the previous results, we obtain:

Theorem (GGPS)
Let G be a unimodular, locally compact group and let Γ a unimodular closed subgroup (e.g. a lattice) such that $X := \Gamma \backslash G$ is **compact**. Then $L^2(X)$ with the natural unitary action of G (by right translation) has a discrete decomposition.

Indeed, we have already seen that T_f is HS on $L^2(X)$, thus compact, so the previous theorem applies.

Application 3: compact groups, Peter-Weyl theory

(I) Consider a compact group K. Let dk be the unique Haar measure with $\int_K dk = 1$.

Theorem Any finite dimensional $V \in \mathrm{Rep}(K)$ has a structure of unitary representation of K, and V is a direct sum of irreducible representations.

Pick any hermitian product $\langle ., . \rangle$ on V and define

$$(v, w) = \int_K \langle k.v, k.w \rangle dk,$$

a K-invariant hermitian product making V unitary. For the second part, if V is irreducible, we are done, otherwise pick a sub-representation $W \neq V$. Then $W^\perp$ is K-stable and $V = W \oplus W^\perp$, so we are done by induction on dim V.

Application 3: compact groups, Peter-Weyl theory

(I) The following theorem is classical for finite groups:

Theorem (Schur's orthogonality relations)
a) Any $V \in \hat{K}$ is finite dimensional.
b) Let $U, V \in \hat{K}$ and let $a_1, a_2 \in U$ and $b_1, b_2 \in V$. Then

$$\int_K \langle k.a_1, a_2 \rangle \overline{\langle k.b_1, b_2 \rangle} dk$$

is 0 when $U \neq V$ and equal to $\frac{\langle a_1, b_1 \rangle \overline{\langle a_2, b_2 \rangle}}{\dim V}$ when $U = V$.

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(II) Let $V \in \hat{K}$. First we prove that there is $d > 0$ such that for all $u, v \in V$ we have

$$\int_K |\langle k.v, w\rangle|^2 = \frac{||v||^2 \bullet ||w||^2}{d}.$$

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$$\int_K |\langle k.v, w\rangle|^2 = \frac{||v||^2 \bullet ||w||^2}{d}.$$

(III) Fix $v_0 \in V$ nonzero. The K-invariant hermitian product

$$(v, w) = \int_K \langle k.v, v_0\rangle \overline{\langle k.w, v_0\rangle} dk$$

is continuous (Cauchy-Schwarz), thus it is given by $\langle Av, w\rangle$ for some $A \in \mathrm{End}_G(V) = \mathbb{C}$ (Schur's lemma). Thus there is $\alpha(v_0) > 0$ such that

$$\int_K |\langle k.v, v_0\rangle|^2 dk = \alpha(v_0)||v||^2, \quad \forall v.$$

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(I) But

$$\int_K |\langle k.v, v_0\rangle|^2 dk = \int_K |\langle v, k^{-1}.v_0\rangle|^2 dk = \int_K |\langle k.v_0, v\rangle|^2 dk.$$

Comparing these formulae yields $\alpha(v) = ||v||^2/d$ for some constant $d > 0$ and proves the first claim.

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Comparing these formulae yields $\alpha(v) = ||v||^2/d$ for some constant $d > 0$ and proves the first claim.

(II) Next we prove that $\dim V < \infty$ and $d = \dim V$. If $e_1, ..., e_n$ is any orthonormal family (not basis a priori!) of V, then $\sum_i |\langle k.v, e_i\rangle|^2 \leq ||k.v||^2 = ||v||^2$ for all v, thus

$$n||v||^2/d = \int_K (\sum_i |\langle k.v, e_i\rangle|^2) dk \leq \int_K ||v||^2 dk = ||v||^2.$$

Application 3: compact groups, Peter-Weyl theory

(I) But

$$\int_K |\langle k.v, v_0\rangle|^2 dk = \int_K |\langle v, k^{-1}.v_0\rangle|^2 dk = \int_K |\langle k.v_0, v\rangle|^2 dk.$$

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(III) Thus $\dim V < \infty$. But then we can choose the e_i an ON-basis of V and then $\sum_i |\langle k.v, e_i\rangle|^2 = ||k.v||^2 = ||v||^2$ for all v. The same computation shows that $n = d$. This finishes the proof for $U = V$.

Application 3: compact groups, Peter-Weyl theory

(I) Suppose now that $U \neq V$. Using the previous results and Cauchy-Schwarz, we deduce that the K-invariant hermitian form

$$B(u, v) = \int_K \langle k.u, a_2 \rangle \overline{\langle k.v, b_2 \rangle} dk$$

is continuous, thus given by $\langle A(u), v \rangle$ for some $A \in \mathrm{Hom}_K(U, V)$. The latter space is 0, so we are done.

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(II) By the previous theorem we can define the character $\chi_\pi \in C(K)$ of $\pi \in \hat{K}$, with $\chi_\pi(k)$ the trace of the endomorphism $v \to k.v$. Define

$$e_\pi = \dim(\pi) \overline{\chi_\pi} \in C(K).$$

Application 3: compact groups, Peter-Weyl theory

(I) For $\pi \in \hat{K}$ and $V \in \mathrm{Rep}(K)$ we can define a continuous linear map

$$T_\pi : V \to V,\, v \to e_\pi.v = \int_K e_\pi(k)k.vdk.$$

If V is a unitary rep. of K, then T_π is a self-adjoint operator on V, since $\overline{e_\pi(g^{-1})} = e_\pi(g)$ (because, by compactness, the eigenvalues of $v \to k.v$ are on the unit circle). We can re-interpret (exercise) the orthogonality relations as follows:

Theorem a) For $\pi \neq \sigma \in \hat{K}$ we have $e_\pi * e_\pi = e_\pi$ and $e_\pi * e_\sigma = 0$. The operator T_π acts by identity on π and by 0 on any other $\sigma \in \hat{K}$.
b) For any $V \in \mathrm{Rep}(G)$, the operator T_π is a projection onto its image $V(\pi)$, called the π-**isotypic component of** V. If V is unitary, T_π is an orthogonal projection.

Application 3: compact groups, Peter-Weyl theory

(I) Consider now $H = L^2(K)$, with the action of K **by left translation** $g.f(x) = f(g^{-1}x)$. Then

$$T_f(\varphi) = f * \varphi, \quad f \in C(K), \varphi \in L^2(K).$$

Theorem (Peter-Weyl)
a) We have canonical isomorphisms $L^2(K)(\pi) \simeq \pi \otimes \pi^*$ for $\pi \in \hat{K}$ and $f = \sum_{\pi \in \hat{K}} e_\pi * f$ for $f \in L^2(K)$.
b) There is a canonical isomorphism of $K \times K$-representations

$$L^2(K) \simeq \widehat{\bigoplus_{\pi \in \hat{K}}} \pi \otimes \pi^*.$$

Application 3: compact groups, Peter-Weyl theory

(I) By GGPS we have a discrete decomposition

$$L^2(K) = \widehat{\bigoplus_{\pi \in \hat{K}}} X_\pi \otimes \pi$$

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(II) Also T_π acts by identity on $X_\pi \otimes \pi$ and by 0 on the other summands, so $L^2(K)(\pi) = \mathrm{Im}(T_\pi) = X_\pi \otimes \pi$. It suffices therefore to prove that $X_\pi \simeq \pi^*$.

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(III) We claim that the inclusion $L^2(K) \subset C(K)$ induces an isomorphism $X_\pi \simeq \mathrm{Hom}_K(\pi, C(K))$. The latter is identified with π^*, by sending $u \in \mathrm{Hom}_K(\pi, C(K))$ to $v \in \pi \mapsto u(v)(1)$ and $l \in \pi^*$ to $v \to (k \to l(k.v))$ (Frobenius reciprocity).

Application 3: compact groups, Peter-Weyl theory

(I) To prove the claim, pick $\varphi \in X_\pi$, we want to prove that $\varphi(\pi) \subset C(K)$. Now $\varphi(\pi)$ is a finite dimensional subspace sub-representation of $L^2(K)$. If $f \in \varphi(\pi)$, then $W = \{T_h(\varphi) | \, h \in C(K)\}$ is finite dimensional (contained in $\varphi(\pi)$) and using Dirac sequences we see that $f \in \overline{W} = W$, thus there is $h \in C(K)$ such that $f = T_h(f) \in C(K)$.

Theorem (Peter-Weyl) For any $V \in \mathrm{Rep}(K)$ the space V_K of K-finite vectors is given by $V_K = \sum_{\pi \in \hat{K}} V(\pi)$ and it is dense in V. There are natural isomorphisms

$$\pi \otimes \mathrm{Hom}_K(\pi, V) \simeq V(\pi).$$

Application 3: compact groups, Peter-Weyl theory

(I) We first prove the inclusion $V_K \subset \sum_\pi V(\pi)$. If $v \in V_K$, then $\mathrm{Span}(K.v)$ is a finite dimensional representation of K, thus a direct sum of irreducible reps. $\pi_1, ..., \pi_n$, and T_π acts by identity on π, thus $v \in \sum V(\pi_i)$.

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(II) For the rest, the crucial claim is that for $v \in V(\pi)$ $W = \overline{\mathrm{Span}(K.v)} \simeq \pi^{\oplus N}$ for some integer $N \geq 1$. It suffices to check that $\dim W < \infty$, since T_π acts by identity on W (and kills any $\sigma \in \hat{K}$ different from π).

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(III) By Hahn-Banach it suffices to check that dim $W^* < \infty$ (continuous dual). But one easily checks that sending $l \in W^*$ to $f_l : k \to l(k^{-1}.v)$ embeds W^* in the finite dimensional space of functions f such that $T_\pi(f) = (f)$ (recall that T_π is compact!).

Application 3: compact groups, Peter-Weyl theory

(I) Next, we prove that V_K is dense in V. If not, pick $l \in V^*$ nonzero but vanishing on V_K. Fix $v \in V$ and set $\varphi(k) = l(k^{-1}.v)$, then $\varphi \in C(K)$ and one checks that $e_\pi * \varphi = 0$ for $\pi \in \hat{K}$, thus by the previous theorem $\varphi = 0$ and $l = 0$.

Theorem (Peter-Weyl) Any irreducible $V \in \mathrm{Rep}(K)$ is finite dimensional.

Each $V(\pi)$ is 0 or V by irreducibility, and $\sum V(\pi)$ is dense, so for some π we have $V(\pi) = V$. But the previous theorem shows that $V(\pi)$ is a direct sum of copies of π, thus $V \simeq \pi$ and we are done.

Problem set

(I) Let H, H' be unitary representations of G (with the usual hypotheses on G), with H irreducible. Prove that any $T \in \mathrm{Hom}_G(H, H')$ has closed image and induces an isomorphism between H and a sub-representation of H'. Hint: use Schur's lemma.

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(II) Let H, H' be unitary representations of G such that $H \simeq H'$ in $\mathrm{Rep}(G)$. Prove that there is an isomorphism $U \in \mathrm{Hom}_G(H, H')$ such that $||U(h)|| = ||h||$ for all $h \in H$.

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(III) Prove that the characters φ_π of elements $\pi \in \hat{K}$ form an ON-basis of $L^2(K)$. Also, a finite dimensional representation V of K is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

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(II) Let $T \in HS(H)$ and let (e_n) and $(f_n)_n$ be an ON-bases of H. Using the Plancherel formula twice, prove that $\sum_n ||T(e_n)||^2 = \sum_n ||T^*(f_n)||^2$. Deduce that $T^* \in HS(H)$ and that $\sum_n ||T(e_n)||^2$ is independent of the ON-basis $(e_n)_n$.

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(III) Prove that any $T \in HS(H)$ is compact. Hint: pick an ON-basis (e_n) and consider the operators $T_n(v) = \sum_{k \leq n} \langle v, e_k \rangle T(e_k)$.

Problem set

(I) Let $T \in B(H)$ and $S \in HS(H)$.
a) Prove that $TS, ST \in HS(H)$.
b) If $T \in HS(H)$, prove that $TS, ST \in TC(H)$.

(II) In this exercise we will prove that any $T \in TC(H)$ can be written $T = AB$ with $A, B \in HS(H)$.

a) Explain why T is compact and why $\ker(T^*T) = \ker(T)$. Deduce that $\ker(T)^\perp$ has an ON-basis $(v_n)_n$ such that $T^*Tv_n = \lambda_n v_n$ for some $\lambda_n > 0$ tending to 0.

b) Define operators S, U by setting them equal to 0 on $\ker(T)$ and asking that $Sv_n = \sqrt[4]{\lambda_n} v_n$ and $Uv_n = \frac{1}{\sqrt{\lambda_n}} v_n$. Prove that $T = US^2$ and that $||Uv|| = ||v||$ for $v \in \ker(T)^\perp$.

c) Let (e_n) be an ON-basis of H such that $\sum ||Te_n|| < \infty$. Prove that $||Te_n|| \geq ||Se_n||^2$ (use Cauchy-Schwarz) and deduce that $S, U \in HS(H)$. Conclude.

Problem set

(I) Let $T \in TC(H)$ and let $(e_n)_n$ and $(f_n)_n$ be two ON-bases of H.
a) Prove that

$$\sum_k |\langle Te_n, f_k\rangle\langle f_k, e_n\rangle| \leq ||Te_n||$$

and deduce that $\sum_{n,k} |\langle Te_n, f_k\rangle\langle f_k, e_n\rangle| < \infty$.
b) By computing $\sum_{n,k} \langle Te_n, f_k\rangle\langle f_k, e_n\rangle$ in two different ways, prove that

$$\sum_n \langle Te_n, e_n\rangle = \sum_n \langle Tf_n, f_n\rangle.$$