Lecture 13: Back to $\mathbb{GL}_2$, game over

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Adelic automorphic forms

(I) **Let G be a connected reductive $\mathbb{Q}$-group, with (complex) Lie algebra $\mathfrak{g}$. Let $\mathfrak{Z}(\mathfrak{g}) = Z(U(\mathfrak{g}))$.** In the adelic world the role of the arithmetic subgroup Γ (resp. of $G(\mathbb{R})$) is played by $G(\mathbb{Q})$ (resp. $G(\mathbb{A})$).

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(II) Pick an embedding (over $\mathbb{Q}$) $G \subset \mathbb{GL}_n(\mathbb{C})$ and define (with $||g_\infty||$ as usual)

$$||g|| = ||g_\infty|| \bullet \prod_p ||g_p||, \;\; ||g_p|| = \max(\max_{ij} |(g_p)_{ij}|_p, 1/|\det(g_p)|_p).$$

This gives a norm on $G(\mathbb{A})$ (depending on the embedding) and a notion (independent of the embedding) of moderate growth for functions $f : G(\mathbb{A}) \to \mathbb{C}$.

Adelic automorphic forms

(I) A map $f : G(\mathbb{A}) \to \mathbb{C}$ is called **smooth** if it is smooth in the "real variable" and locally constant in the "finite variable", via the decomposition $G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f)$, i.e. for any $g = (g_\infty, g_f) \in G(\mathbb{A})$ there is a neighborhood $V = V_\infty \times V_f$ of g and $\varphi \in C^\infty(V_\infty)$ such that

$$f(x_\infty, x_f) = \varphi(x_\infty), \ (x_\infty, x_f) \in V.$$

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$$f(x_\infty, x_f) = \varphi(x_\infty), \ (x_\infty, x_f) \in V.$$

(II) A map $f \in C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is an automorphic form for G if

- f is K_∞-finite and invariant by right translation by some compact open subgroup K_f of $G(\mathbb{A}_f)$.

- f is $\mathfrak{Z}(\mathfrak{g})$-finite.

- f has moderate growth.

Adelic automorphic forms

(I) The space $\mathscr{A}(G)$ of adelic automorphic forms for G has a natural action of $G(\mathbb{A}_f)$, by right translation. If K_f is a compact open subgroup of $G(\mathbb{A}_f)$ let

$$\mathscr{A}(G, K_f) = \mathscr{A}(G)^{K_f}.$$

By definition

$$\mathscr{A}(G) = \varinjlim_{K_f} \mathscr{A}(G, K_f).$$

Adelic automorphic forms

(I) We can relate each $\mathscr{A}(G, K_f)$ with a space of classical automorphic forms for various congruence subgroups of $G(\mathbb{Q})$, depending on K_f. The finiteness of the class number of G ensures that one can write

$$G(\mathbb{A}_f) = \coprod_{i=1}^{h} G(\mathbb{Q}) g_i K_f.$$

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(II) Letting $\Gamma_i = G(\mathbb{Q}) \cap g_i K_f g_i^{-1}$, we obtain a homeomorphism

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_f \simeq \coprod_{i=1}^{h} \Gamma_i \backslash G(\mathbb{R}), \Gamma_i x \to G(\mathbb{Q})(x, g_i) K_f.$$

Unwinding definitions, we obtain

$$\mathscr{A}(G, K_f) \simeq \bigoplus_{i=1}^{h} \mathscr{A}(G, \Gamma_i),\ f \to (x \to f(x, g_i)).$$

It also follows that $\mathscr{A}(G)$ is a $(\mathfrak{g}, K) \times G(\mathbb{A}_f)$-module.

A useful reduction

(I) Let A_G be the split component of G and

$$G(\mathbb{A})^1 = \{g \in G(\mathbb{A}) | \, |\chi(g)| = 1 \ \forall \chi \in X(G)_{\mathbb{Q}}\}.$$

The adelic analogue of the decomposition $G(\mathbb{R}) =^0 G(\mathbb{R}) \times A_G$ is

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(II) The **automorphic quotient**

$$[G] = G(\mathbb{Q})A_G \backslash G(\mathbb{A}) \simeq G(\mathbb{Q}) \backslash G(\mathbb{A})^1$$

has finite invariant measure (Borel, Harish-Chandra) and the study of $\mathscr{A}(G)$ reduces to that of

$$\mathscr{A}(G)^1 := \{f \in \mathscr{A}(G) | \, f(zx) = f(x), \; z \in A_G, x \in G(\mathbb{A})\}.$$

A useful reduction

(I) More precisely

- there is a surjective homomorphism with kernel $G(\mathbb{A})^1$

$$H_G : G(\mathbb{A}) \to \mathrm{Hom}_{\mathbb{Z}}(X(G)_{\mathbb{Q}}, \mathbb{R}), \;\; H_G(g)(\chi) := \log|\chi(g)|,$$

where $|\bullet| : \mathbb{A}^* \to \mathbb{R}_{>0}$ is the usual character.

- for any $\lambda \in \mathfrak{a}_G^* \otimes \mathbb{C}$ and any polynomial function P on $\mathfrak{a}_G$ the map

$$f_{\lambda,P}(g) = e^{H_G(g)(\lambda)} p(H_G(g))$$

is in $\mathscr{A}(G)$ (exercise!). Let Pol be the vector space generated by these functions as λ and P vary.

- the multiplication map induces an isomorphism

$$\mathrm{Pol} \otimes_{\mathbb{C}} \mathscr{A}(G)^1 \simeq \mathscr{A}(G).$$

Cusp forms

(I) If P is a $\mathbb{Q}$-parabolic of G, with unipotent radical N, then $N(\mathbb{Q})\backslash N(\mathbb{A})$ is compact and for any $f \in C(G(\mathbb{Q})\backslash G(\mathbb{A}))$ we can define its constant term along P by

$$f_P(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(ng)dn,\ g \in G(\mathbb{A}).$$

The automorphic form f is called **cuspidal or cusp form** if its constant terms along **proper** $\mathbb{Q}$-parabolics of G vanish.

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(II) The space $\mathscr{A}(G)^1_{\mathrm{cusp}}$ of cusp forms in $\mathscr{A}(G)^1$ is a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-submodule of $\mathscr{A}(G)$.

Cusp forms

(I) Let

$$\mathscr{A}(G)^1_{L^2} = \{f \in \mathscr{A}(G)^1 | \int_{[G]} |f(x)|^2 dx < \infty\}.$$

The classical-adelic dictionnary and the classical GPS theorem give

Theorem (Gelfand, Piatetski-Shapiro)

a) Any $f \in \mathscr{A}(G)^1_{\mathrm{cusp}}$ is bounded.

b) The $G(\mathbb{A})^1$-representation $L^2([G])_{\mathrm{cusp}}$ has a discrete decomposition.

c) $\mathscr{A}(G)^1_{\mathrm{cusp}}$ and $\mathscr{A}(G)^1_{L^2}$ are semi-simple $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-modules, with finite multiplicities.

The discrete spectrum

(I) The irreducible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-modules appearing in the decomposition of $\mathscr{A}(G)^1_{\mathrm{cusp}}$ are called **cuspidal automorphic representations** of $G(\mathbb{A})^1$. Note that they are not really representations of $G(\mathbb{A})$, only of $G(\mathbb{A}_f)$!

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(II) Let $L^2([G])_{\mathrm{disc}}$ be the Hilbert sum of all irreducible sub-representations of $L^2([G])$. This is called the **discrete spectrum** and by the above theorem it contains the cuspidal part. These two are equal if and only if $[G]$ is compact (the constant function 1 is not cuspidal, but belongs to the discrete spectrum when the quotient is not compact).

The discrete spectrum

(I) The next result is a beautiful and important reformulation of the finiteness theorem:

Theorem (Harish-Chandra) $L^2([G])_{\mathrm{disc}}$ has a discrete decomposition, i.e. for any $\pi \in \widehat{G(\mathbb{A})}$ we have

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$$\dim \mathrm{Hom}_{G(\mathbb{A})}(\pi, L^2([G])) < \infty.$$

(II) By the classical-adelic dictionary and simple manipulations one reduces this to: if $A_G = \{1\}$ and $\Gamma \subset G(\mathbb{Q})$ is arithmetic, then for any $\pi \in \widehat{G(\mathbb{R})}$

$$\dim \mathrm{Hom}_{G(\mathbb{R})}(\pi, L^2(\Gamma \backslash G(\mathbb{R}))) < \infty.$$

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(III) First, by Segal's theorem π^∞ is killed by a codimension 1 ideal J of $\mathfrak{Z}$.

The discrete spectrum

(I) Next, pick $v \in HC(\pi)$, WLOG $v \in \pi(\sigma)$ for some $\sigma \in \widehat{K_\infty}$. Then evaluation at v gives an embedding

$$\mathrm{Hom}_{G(\mathbb{R})}(\pi, L^2(\Gamma \backslash G(\mathbb{R}))) \subset \mathscr{A}(G, \Gamma)[J, \sigma],$$

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(II) The key point in proving the embedding above is the following: if $\varphi \in \mathrm{Hom}_{G(\mathbb{R})}(\pi, L^2(\Gamma\backslash G(\mathbb{R})))$ and $f = \varphi(v)$, then f is clearly of type J, σ, and we need to show that f has moderate growth. But $f \in L^2(\Gamma\backslash G(\mathbb{R})) \subset L^1(\Gamma\backslash G(\mathbb{R}))$, so we are done by the first fundamental estimate.

Smooth representations, factorisation theorem

(I) **From now on we take** $G = \mathbb{GL}_2$ **and write**

$$G_v = G(\mathbb{Q}_v), \; K_\infty = O(2), \; K_p = G(\mathbb{Z}_p).$$

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(II) The group $G(\mathbb{A}_f)$ is "essentially" (but not really) the product of the various G_p, so it is not unreasonable to think that its irreducible representations are obtained from irreducible representations of the various G_p. For this we have to be more precise about which representations we consider.

Smooth representations of locally profinite groups

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(II) Let

$$\mathrm{Rep}^{\mathrm{sm}}(\mathscr{G})$$

be the category of **smooth representations** of $\mathscr{G}$, i.e. $\mathbb{C}$-linear abstract representations π of $\mathscr{G}$ such that

$$\pi = \bigcup_{K \leq \mathscr{G}} \pi^K,$$

the union being taken over compact open subgroups K of $\mathscr{G}$. Equivalently, the stabiliser of any vector in π is open.

Smooth representations of locally profinite groups

(I) A representation $\pi \in \mathrm{Rep}^{\mathrm{sm}}(\mathscr{G})$ is called **admissible** if $\dim \pi^K < \infty$ for any compact open subgroup K of $\mathscr{G}$, or equivalently $\mathrm{Hom}_K(\sigma, \pi)$ is finite dimensional for any $\sigma \in \hat{K}$.

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(II) The following result is the p-adic analogue of the admissibility of irreducible $(\mathfrak{g}, K)$-modules seen in the last lecture. It is true for any p-adic reductive group, not only G_p, but the proof is quite difficult (already for G_p):

Theorem (Bernstein, Jacquet) Any irreducible $\pi \in \mathrm{Rep}^{\mathrm{sm}}(G_p)$ is admissible.

Hecke algebras

(I) The **Hecke algebra** of $\mathscr{G}$ is the space

$$\mathscr{H}(\mathscr{G}) = LC_c(\mathscr{G})$$

of locally constant functions $f : \mathscr{G} \to \mathbb{C}$ with compact support, endowed with the convolution product

$$f * g(x) = \int_{\mathscr{G}} f(u)g(u^{-1}x)du,$$

where we fix a Haar measure du on $\mathscr{G}$. It is a non-unital associative algebra.

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where we fix a Haar measure du on $\mathscr{G}$. It is a non-unital associative algebra.

(II) Any $\pi \in \mathrm{Rep}^{\mathrm{sm}}(\mathscr{G})$ is naturally a module over $\mathscr{H}(\mathscr{G})$, via

$$f.v = \int_{\mathscr{G}} f(g) g.v dg = \mathrm{vol}(K) \sum_{g \in \mathscr{G}/K} f(g) g.v,$$

where K is a sufficiently small compact open subgroup of $\mathscr{G}$. The sum is finite since f has compact support.

Hecke algebras

(I) One shows without too much difficulty that $\mathrm{Rep}^{\mathrm{sm}}(\mathscr{G})$ is equivalent to the category of $\mathscr{H}(\mathscr{G})$-modules M such that any $m \in M$ satisfies $f.m = m$ for some $f \in \mathscr{H}(\mathscr{G})$.

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(II) If K is a compact open subgroup of $\mathscr{G}$ let

$$\mathscr{H}(\mathscr{G}, K) = \{f \in \mathscr{H}(\mathscr{G}) | \, f(k_1 g k_2) = f(g), k_1, k_2 \in K, g \in \mathscr{G}\}.$$

This is a sub-algebra of $\mathscr{H}(\mathscr{G})$, having $\frac{1}{\mathrm{vol}(K)} 1_K$ as a unit element. Moreover

$$\mathscr{H}(\mathscr{G}) = \varinjlim_{K} \mathscr{H}(\mathscr{G}, K).$$

For any $\pi \in \mathrm{Rep}^{\mathrm{sm}}(\mathscr{G})$ the space π^K is naturally a module over $\mathscr{H}(\mathscr{G}, K)$, and if π is irreducible and $\pi^K \neq 0$, this module is simple (excellent exercise).

Hecke algebras

(I) A crucial example for the sequel is the case $\mathscr{G} = G_p$ and $K = K_p$. Normalize dg so that $\mathrm{vol}(K_p) = 1$. The algebra $\mathscr{H}(G_p, K_p)$ is called the **spherical Hecke algebra**. It has a very beautiful description:

Theorem There is a natural isomorphism of $\mathbb{C}$-algebras

$$\mathscr{S} : \mathbb{C}[X^{\pm 1}, Y] \simeq \mathscr{H}_p.$$

As $\mathbb{C}$-vector spaces

$$\mathscr{H}_p = \bigoplus_{g \in K_p \backslash G_p / K_p} \mathbb{C} 1_{K_p g K_p},$$

and by elementary divisors

$$G_p = \coprod_{a \leq b \in \mathbb{Z}} K_p \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix} K_p.$$

Hecke algebras

(I) Thus a $\mathbb{C}$-basis of $\mathscr{H}_p$ is given by the functions

$$\varphi_{a,b} = 1_{K_p \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix} K_p}.$$

Let

$$R_p = \varphi_{1,1}, \;\; T_p = \varphi_{0,1}.$$

One can check by hand that sending X to R_p and Y to T_p yields the isomorphism in the theorem.

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(II) More canonically and closer to the situation for real groups, we have an analogue of the Harish-Chandra transform for $\mathbb{SL}_2(\mathbb{R})$, the **Satake transform** (for $\varphi \in \mathscr{H}_p$)

$$S(\varphi)(t) = |a/d|_p^{1/2} \int_{\mathbb{Q}_p} \varphi(t \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}) dx, \;\; t = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

Hecke algebras

(I) One proves that it induces an isomorphism of $\mathbb{C}$-algebras (the **Satake isomorphism**)

$$\mathscr{H}_p \simeq \mathrm{LC}_c(T(\mathbb{Q}_p)/T(\mathbb{Z}_p))^{S_2} \simeq$$

$$\simeq \mathbb{C}[X^{\pm}, Y^{\pm}]^{S_2} \simeq \mathbb{C}[X+Y, (XY)^{\pm}].$$

where T is the diagonal torus and $S_2 \simeq \mathbb{Z}/2\mathbb{Z}$ (Weyl group of (G, T)) acts by permuting the diagonal entries. This isomorphism sends R_p to XY and T_p to $\sqrt{p}(X+Y)$.

Spherical representations

(I) An irreducible representation $\pi \in \mathrm{Rep}^{\mathrm{sm}}(G_p)$ is called **unramified or spherical** if $\pi^{K_p} \neq 0$. In this case π^{K_p} is a simple module over $\mathscr{H}_p \simeq \mathbb{C}[X^{\pm 1}, Y]$, thus it is one-dimensional, and $T_p, R_p \in \mathscr{H}_p$ act on $\pi_p^{K_p}$ by scalars $T_p(\pi)$ and $R_p(\pi)$. Moreover π is determined up to isomorphism by these scalars.

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(II) The **Satake parameters** of π are the roots of the polynomial $X^2 - p^{-1/2} T_p(\pi) X + R_p(\pi)$. They form an un-ordered pair $(t_1, t_2) \in (\mathbb{C}^* \times \mathbb{C}^*)/S_2$, and they determine the spherical representation π up to isomorphism. Conversely, any un-ordered pair arises from a spherical representation, thus

$$\{\text{spherical representations of } G_p\}/\simeq \longleftrightarrow (\mathbb{C}^* \times \mathbb{C}^*)/S_2.$$

Spherical representations

(I) More precisely, given $t_1, t_2 \in \mathbb{C}^*$ consider the unramified characters

$$\chi_i : \mathbb{Q}_p^* \to \mathbb{C}^*, \chi_i(x) = t_i^{v_p(x)}$$

and the induced representation

$$I(\chi_1, \chi_2) = \{\varphi \in LC(G_p) | \, \varphi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g)$$

$$= \chi_1(a)\chi_2(d)|a/d|_p^{1/2}\varphi(g) \ \forall a, b, d, g\}$$

with $g.\varphi(x) = \varphi(xg)$.

Spherical representations

(I) Then one can check that $I(\chi_1, \chi_2)$ has a unique spherical sub-quotient $\pi(t_1, t_2)$, whose isomorphism class depends only on the set $\{t_1, t_2\}$. Moreover, $\dim \pi(t_1, t_2) < \infty$ if and only if $t_1/t_2 \in \{p, p^{-1}\}$, in which case $\dim \pi(t_1, t_2) = 1$.

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(II) Conversely, if π is unramified, with Satake parameters t_1, t_2, then

$$\pi \simeq \pi(t_1, t_2).$$

Spherical representations

(I) Note that $I(\chi_1, \chi_2)$ makes sense for any smooth (i.e. open kernel) characters $\chi_i : \mathbb{Q}_p^* \to \mathbb{C}^*$. It will be proved in Olivier Taibi's course that $I(\chi_1, \chi_2)$ is admissible, of length at most 2, irreducible when $\chi_1\chi_2^{-1} \neq |\bullet|_p^{\pm 1}$. Moreover $I(\chi_1, \chi_2)$ has a unique infinite-dimensional sub-quotient $\pi(\chi_1, \chi_2)$, and

$$\pi(\chi_1, \chi_2) \simeq \pi(\delta_1, \delta_2) \Leftrightarrow (\delta_1, \delta_2) \in \{(\chi_1, \chi_2), (\chi_2, \chi_1)\}.$$

The factorisation theorem

(I) Let π_∞ be an irreducible $(\mathfrak{g}, K_\infty)$-module and let π_p be a smooth irreducible representation of G_p, such that π_p is spherical for almost all p (i.e. all but finitely many).

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(II) Fix nonzero vectors $e_p \in \pi_p^{K_p}$ for all p for which π_p is spherical and define the **restricted tensor product**

$$\otimes'_v \pi_v := \varinjlim_S \otimes_{v\in S}\pi_v,$$

over all finite subsets S of $\{2, 3, 5, ...\} \cup \{\infty\}$ containing ∞ and all those p for which π_p is not spherical, the transition maps being $\otimes_{v\in S} x_v \to \otimes_{v\in S} x_v \otimes \otimes_{v\in S'\setminus S} e_v$ for $S \subset S'$.

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(III) Somewhat more concretely $\otimes'_v \pi_v$ is spanned by vectors of the form $\otimes_v x_v$ with $x_v \in \pi_v$ and $x_p = e_p$ for almost all p.

The factorisation theorem

(I) We then have the following fundamental local-global result:

Theorem (Flath's factorisation theorem) a) $\otimes'_v \pi_v$ is an irreducible, smooth and admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module, independent up to isomorphism on the choice of e_p.

b) Any irreducible, smooth and admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module is obtained by this construction, and the local factors π_v are uniquely determined up to isomorphism.

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b) Any irreducible, smooth and admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module is obtained by this construction, and the local factors π_v are uniquely determined up to isomorphism.

(II) There is also a "topological" version of the above algebraic theorem, which is much harder to prove. Namely, consider now $\pi_v \in \widehat{G_v}$, almost all of them being spherical (same definition as in the algebraic case).

The factorisation theorem

(I) This gives rise to a unitary representation $\pi = \widehat{\otimes}' \pi_v$ of $G(\mathbb{A})$, completion of $\otimes' \pi_v$ (defined as above, with e_p chosen of norm 1) for the hermitian product

$$\langle \otimes x_v, \otimes y_v \rangle = \prod_v \langle x_v, y_v \rangle.$$

Theorem (Bernstein, Flath) We have $\widehat{\otimes}'_v \pi_v \in \widehat{G(\mathbb{A})}$ (and independent, up to isomorphism, of the choice of the unitary spherical vectors e_p) and any $\pi \in \widehat{G(\mathbb{A})}$ is obtained this way, the local factors π_v being uniquely determined up to isomorphism.

The factorisation theorem

(I) The two theorems are closely related: if $\Pi \in \widehat{G(\mathbb{A})}$ has local factors Π_v and if

$$\pi_\infty = HC(\Pi_\infty) = \Pi_\infty^{K_\infty-\mathrm{fin}}, \ \pi_p = \Pi_p^{\mathrm{sm}} := \bigcup_{K \leq G_p} \Pi_p^K,$$

then $\pi_p \in \mathrm{Rep}(G_p)^{\mathrm{sm}}$ is irreducible, π_∞ is an irreducible $(\mathfrak{g}, K_\infty)$-module (cf. previous lecture) and

$$\Pi^{K-\mathrm{fin}} \simeq \otimes' \pi_v$$

as $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-modules, where $K = K_\infty \times \prod_p K_p$.

The case of modular forms

(I) Let now $N \geq 1$ be an integer and consider $f \in S_k(N) = S_k(\Gamma_0(N))$, say with $k \geq 2$. We saw that we can attach to f an automorphic form on $\Gamma_0(N)\backslash\mathbb{SL}_2(\mathbb{R})$. Now, a simple exercise shows that there is a natural homeomorphism

$$\Gamma_0(N)\backslash\mathbb{SL}_2(\mathbb{R}) \simeq Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})/K_0(N),$$

where Z is the center of G and

$$K_0(N) = \{g \in G(\hat{\mathbb{Z}}) | \, g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N}\} =$$

$$\prod_{p|N} \mathrm{Iw}_p^N \times \prod_{\gcd(p,N)=1} K_p,$$

with

$$\mathrm{Iw}_p^N = \{g \in K_p | \; g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{p^{v_p(N)}}\}.$$

The case of modular forms

(I) This induces an embedding

$$S_k(N) \to \mathscr{A}(G)_{\mathrm{cusp}}, f \to \varphi_f$$

with image consisting of those $\varphi \in \mathscr{A}(G)_{\mathrm{cusp}}$ killed by $\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \in U(\mathfrak{g})$, right $K_0(N)$-invariant and such that

$$\varphi(g \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}) = e^{ik\theta}\varphi(g).$$

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(II) The construction $f \to \varphi_f$ is compatible with the natural inner products: for a suitable Haar measure dg on $G(\mathbb{A})$ we have

$$\int_{\Gamma_0(N)\backslash\mathscr{H}} |f(z)|^2 y^k \frac{dxdy}{y^2} = \int_{G(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})} |\varphi_f(g)|^2 dg.$$

The case of modular forms

(I) Since φ_f is right $K_0(N)$-invariant, it follows that for $\gcd(p, N) = 1$ the map φ_f is right K_p-invariant. A direct computation shows that

$$\varphi_{T_p(f)} = p^{\frac{k}{2}-1} T_p.\varphi_f,$$

where $T_p.\varphi_f$ is the action of $T_p \in \mathscr{H}_p$ on φ_f.

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where $T_p.\varphi_f$ is the action of $T_p \in \mathscr{H}_p$ on φ_f.

(II) Let

$$\pi(f) = \overline{\mathbb{C}[G(\mathbb{A})]\varphi_f} \subset L^2([G])_{\mathrm{cusp}}.$$

The case of modular forms

(I) The next result is quite deep:

Theorem $\pi(f)$ is irreducible if and only if f is an eigenvector of all T_p with $\gcd(p, N) = 1$. Moreover, if $f, f' \in S_k(N)$ are $\mathbb{T}^{(N)}$-eigenforms, then $\pi(f) = \pi(f')$ if and only if the eigenvalues of T_p on f and f' are the same for almost all p.

Let us focus only on the first part. One implication is easy: if $\pi(f)$ is irreducible, by the factorisation theorem it is a restricted tensor product of local factors π_v. But $\pi(f)^{K_p} \neq 0$ for $\gcd(p, N) = 1$, thus π_p must be spherical for these p, and thus $\mathscr{H}(G_p, K_p)$ acts by scalars on $\pi_p^{K_p}$, thus also on $\pi(f)^{K_p}$, and thus on φ_f itself. But then T_p acts by a scalar on f by the compatibility of $f \to \varphi_f$ with Hecke operators.

The case of modular forms

(I) The other implication is much deeper. Say $T_p(f) = \lambda_p f$ for $\gcd(p, N) = 1$. Then $T_p.\varphi_f = p^{1-\frac{k}{2}}\lambda_p\varphi_f$ and $R_p.\varphi_f = \varphi_f$. Let Π be an irreducible summand of $\Pi(f) = \pi(f)^{K-\mathrm{fin}} \subset \mathscr{A}(G)_{\mathrm{cusp}}$.

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(II) Let F be the projection of φ_f on Π. Clearly $F \neq 0$ (as φ_f generates $\Pi(f)$), F is K_p-invariant and $T_p.F = p^{1-\frac{k}{2}}\lambda_p F$. Thus if Π_v are the local factors of Π, Π_p is spherical with Satake parameters t_1, t_1 satisfying $p^{1/2}(t_1 + t_2) = p^{1-k/2}\lambda_p$ and $t_1 t_2 = 1$. It follows that the local factors at any p prime to N of any irreducible summand of $\Pi(f)$ are isomorphic.

The case of modular forms

(I) The result follows then from the next deep theorem, which will hopefully be seen in Olivier Taibi's course. It is due to the work of many people: Jacquet-Langlands, Piatetski-Shapiro, Miyake, Casselman, etc:

Theorem (strong multiplicity one) Let $\Pi, \Pi' \subset \mathscr{A}(G)_{\mathrm{cusp}}$ be irreducible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-submodules such that the local factors Π_v and Π'_v are isomorphic for all but finitely many places v. Then $\Pi = \Pi'$.

In particular this implies that

$$\dim \mathrm{Hom}_{G(\mathbb{A})}(\pi, L^2([G])_{\mathrm{cusp}}) \leq 1$$

for all $\pi \in \widehat{G(\mathbb{A})}$, a result known as the multiplicity one theorem.

The case of modular forms

(I) Say $f \in S_k(N)$ satisfies $T_p(f) = \lambda_p f$ for $\gcd(p, N) = 1$. If π_p are the local factors of $\pi(f)$, then π_p is spherical for $\gcd(p, N) = 1$, with Satake parameters

$$t_{1,p} = p^{\frac{1-k}{2}}\alpha_p, \;\; t_{2,p} = p^{\frac{1-k}{2}}\beta_p,$$

where

$$X^2 - \lambda_p X + p^{k-1} = (X - \alpha_p)(X - \beta_p).$$

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(II) The next theorem, the Ramanujan-Petersson conjecture for modular forms is a very deep and difficult result.

Theorem (Deligne) If $f \in S_k(N)$ satisfies $T_p(f) = \lambda_p f$ for $\gcd(p, N) = 1$, then the Satake parameters of $\pi_p(f)$ for $\gcd(p, N) = 1$ have absolute value 1, and so

$$|\lambda_p| \leq 2p^{\frac{k-1}{2}}.$$