Lecture 12: Harish-Chandra's world...

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$(\mathfrak{g}, K)$-modules

(I) Let G be a connected reductive group defined over $\mathbb{R}$ and let K be a maximal compact subgroup of $G(\mathbb{R})$. Let

$$\mathfrak{g} = \mathrm{Lie}(G),\ \mathfrak{g}_{\mathbb{R}} := \mathrm{Lie}(G(\mathbb{R})),\ \mathfrak{k} := \mathrm{Lie}(K),$$

so that $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

$(\mathfrak{g}, K)$-modules

(I) Recall that a $(\mathfrak{g}, K)$-**module** is a $\mathbb{C}$-vector space M (**no topology**!) together with $\mathbb{C}$-linear actions (of Lie algebras, resp. groups) of $\mathfrak{g}$ and of K, such that

- for any $m \in M$ the space $\mathbb{C}[K]m$ is finite dimensional and affords a continuous (thus smooth) action of K.

- For $X \in \mathfrak{k}$ and $m \in M$ we have

$$X.m = \lim_{t \to 0} \frac{\exp(tX).m - m}{t}.$$

- For $X \in \mathfrak{g}$, $k \in K$ and $m \in M$ we have

$$k.(X.(k^{-1}.m)) = \mathrm{Ad}(k)(X).m.$$

Let $(\mathfrak{g}, K) - Mod$ be the category of $(\mathfrak{g}, K)$-modules.

The enveloping algebra

(I) The category of $\mathfrak{g}$-representations is equivalent to that of left $U(\mathfrak{g})$-modules. A classical but nontrivial result:

Theorem (Poincaré-Birkhoff-Witt) If $X_1, \ldots, X_n$ is a $\mathbb{C}$-basis of $\mathfrak{g}$, then the monomials $X_1^{k_1} \ldots X_n^{k_n}$ (with $k_i \in \mathbb{Z}_{\geq 0}$) form a $\mathbb{C}$-basis of $U(\mathfrak{g})$.

In particular $U(\mathfrak{g})$ has countable dimension over $\mathbb{C}$. We give next a very nice application of this observation.

Dixmier's Schur lemma

(I) Let $Z(\mathfrak{g})$ be the centre of $U(\mathfrak{g})$. We will see later on that any $D \in Z(\mathfrak{g})$ commutes with $G(\mathbb{R})$, thus acts by endomorphisms on any $(\mathfrak{g}, K)$-module and on V^∞ for any $V \in \mathrm{Rep}(G(\mathbb{R}))$. The analogue of Schur's lemma in $\mathrm{Rep}(G(\mathbb{R}))$ for $(\mathfrak{g}, K) - \mathit{Mod}$ is:

Theorem (Dixmier) If $M \in (\mathfrak{g}, K) - \mathit{Mod}$ is a simple object, then $\mathrm{End}_{(\mathfrak{g},K)-\mathit{Mod}}(M) = \mathbb{C}$. In particular $Z(\mathfrak{g})$ acts by scalars on M.

The same result (with the same proof) applies to simple $U(\mathfrak{g})$-modules.

Dixmier's Schur lemma

(I) Let T be a non scalar endomorphism. By simplicity $T - a$ is invertible for $a \in \mathbb{C}$, thus $P(T)$ is invertible for $P \in \mathbb{C}[X]$ nonzero. Thus $\mathbb{C}(X)$ embeds (as $\mathbb{C}$-vector space) in $\mathrm{End}(M)$ and $\dim_{\mathbb{C}} \mathrm{End}(M)$ is uncountable.

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(II) Let $v \in M \setminus \{0\}$, then again by simplicity $f \to f(v)$ induces an embedding

$$\mathrm{End}(M) \subset M.$$

On the other hand $U(\mathfrak{g})\mathbb{C}[K]v$ is a nonzero sub-$(\mathfrak{g}, K)$-module of M, thus equal to M. Since $U(\mathfrak{g})$ has countable dimension, so does M, contradicting the previous paragraph!

Segal's Schur lemma

(I) The next result is much more subtle.

Theorem (Segal) Let V be an irreducible unitary representation of $G(\mathbb{R})$. Then $Z(\mathfrak{g})$ acts by scalars on V^∞.

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(II) Let $(.,.)$ be the $G(\mathbb{R})$-invariant inner product on V. A simple computation shows that $(Xv, w) = -(v, Xw)$ for $X \in \mathfrak{g}_\mathbb{R}$ and $v, w \in V^\infty$. The map $X + iY \in \mathfrak{g} \to -(X - iY) \in \mathfrak{g}$ extends to a semi-linear anti-automorphism $U(\mathfrak{g}) \to U(\mathfrak{g}), D \to D^\vee$, preserving $Z(\mathfrak{g})$ and such that $(Dv, w) = (v, D^\vee w)$ for $v, w \in V^\infty$ and $D \in U(\mathfrak{g})$.

Segal's Schur lemma

(I) Now let $D \in Z(\mathfrak{g})$ and suppose that for some $v \in V^\infty$ we have $Dv \notin \mathbb{C}v$. We will prove below that for any $x, y \in V^\infty$ there is a sequence $f_n \in C_c^\infty(G(\mathbb{R}))$ such that $f_n.v \to x$ and $f_n Dv \to y$. Then for any $z \in V^\infty$

$$(y, z) = \lim_{n\to\infty} (f_n Dv, z) = \lim_{n\to\infty} (Df_n v, z) =$$

$$\lim_{n\to\infty} (f_n v, D^\vee z) = (x, D^\vee z) = (Dx, z),$$

where we used that D and f_n commute since $D \in Z(\mathfrak{g})$ must commute with the adjoint action of $G(\mathbb{R})$ (cf. next slides). Since V^∞ is dense, it follows that $y = Dx$ for any $x, y \in V^\infty$, a contradiction.

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(II) Thus, to finish the proof, it suffices to prove that for any linearly independent family $v_1, ..., v_n \in V^\infty$ the set $Y := \{(f.v_1, ..., f.v_n) | \, f \in C_c^\infty(G(\mathbb{R}))\}$ is dense in V^n.

Segal's Schur lemma

(I) Let X be the closure of Y. One easily checks that Y is $G(\mathbb{R})$-stable, thus so is X. It easily follows that the orthogonal projection $p : V^n \to X$ is $G(\mathbb{R})$-equivariant. But by Schur's lemma $\mathrm{End}_{G(\mathbb{R})}(V^n) = M_n(\mathbb{C})$, thus $p(x) = Ax$ for some $A \in M_n(\mathbb{C})$. But $(v_1, \ldots, v_n) \in X$ (use a Dirac sequence), so $p(v_1, \ldots, v_n) = (v_1, \ldots, v_n)$. Since the v_i are linearly independent over $\mathbb{C}$, this forces $A = I$ and $p = \mathrm{id}$, thus $X = V^n$ and we are done.

Application of elliptic regularity

(I) The rest of the lecture is devoted to proving that $Z(\mathfrak{g})$ has a huge influence on the representation theory of $G(\mathbb{R})$. We will need the following nontrivial consequence of the elliptic regularity theorem, which we take for granted:

Theorem Let $V \in \mathrm{Rep}(G(\mathbb{R}))$ and let $v \in HC(V)$ be a $Z(\mathfrak{g})$-finite vector. Then for any $l \in V^*$ the map $G(\mathbb{R}) \to \mathbb{C}, g \to l(g.v)$ is real analytic.

Admissibility

(I) For any $M \in (\mathfrak{g}, K) - Mod$ we have

$$M = \bigoplus_{\pi \in \hat{K}} M(\pi),$$

where $M(\pi)$ is the π-isotypic component of M, i.e. $M(\pi) = e_\pi(M)$, where e_π is the idempotent associated to π. Equivalently, $M(\pi)$ is the sum of all K-subrepresentations of M isomorphic to π.

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(II) We say that M is **admissible** if $M(\pi)$ is finite dimensional for all $\pi \in \hat{K}$. The Harish-Chandra functor preserves admissibility

$$HC : \mathrm{Rep}(G(\mathbb{R})) \to (\mathfrak{g}, K) - Mod, \ \ HC(V) := V^{K-\mathrm{fin}} \cap V^\infty.$$

Admissibility

(I) Here is a first crucial result, making the theory of admissible representations of $G(\mathbb{R})$ essentially algebraic:

Theorem (Harish-Chandra) Let $V \in \mathrm{Rep}(G(\mathbb{R}))$ be an admissible representation.

a) We have $HC(V) = V^{K-\mathrm{fin}}$ and any $v \in HC(V)$ is $Z(\mathfrak{g})$-finite.

b) The maps $W \to HC(W)$ and $N \to \bar{N}$ give a bijection between sub-representations of V and sub-objects of $HC(V)$. In particular V is irreducible if and only if $HC(V)$ is so.

With similar arguments one proves that if V, W are admissible $G(\mathbb{R})$-representations, then any continuous linear map $f : V \to W$ which sends $HC(V)$ to $HC(W)$ and is $(\mathfrak{g}, K)$-equivariant is actually $G(\mathbb{R})$-equivariant.

Admissibility

(I) We start by proving that $HC(V) = V^{K-\mathrm{fin}}$. Since $V^{K-\mathrm{fin}} = \oplus_\pi V(\pi)$ (cf. lecture 2), it suffices to show that $V(\pi) \subset V^\infty$ for $\pi \in \hat{K}$. Since $V(\pi)$ is finite dimensional by assumption, this reduces further to the density of $V(\pi) \cap V^\infty$ in $V(\pi)$.

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(II) Pick $v \in V(\pi)$ and f_n a Dirac sequence consisting of smooth functions. Extend $e_\pi \in C(K)$ to $C(G(\mathbb{R}))$ and consider $e_\pi.(f_n.v) = (e_\pi * f_n).v$. These vectors are in $V(\pi) \cap V^\infty$ and converge to $e_\pi.v = v$, so we are done.

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(III) Since $V(\pi)$ is finite dimensional and preserved by $Z(\mathfrak{g})$, it is clear that it consists of $Z(\mathfrak{g})$-finite vectors, thus so does $HC(V)$.

Admissibility

(I) We next show that if N is $(\mathfrak{g}, K)$-stable in $M := HC(V)$, then $\bar{N}$ is $G(\mathbb{R})$-stable. Since $G(\mathbb{R}) = G(\mathbb{R})^0 K$ by the Cartan decomposition, it suffices to check that $G(\mathbb{R})^0 N \subset \bar{N}$.

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(II) By Hahn-Banach it suffices to check that any $l \in V^*$ vanishing on $\bar{N}$ also vanishes on $G(\mathbb{R})^0 N$. By the previous theorem for any $v \in N$ the map $f : g \to l(gv)$ is real analytic on $G(\mathbb{R})^0$. Its derivatives at 1 are computed in terms of the action of $U(\mathfrak{g})$ on v, and l vanishes on $U(\mathfrak{g})v$, thus all derivatives at 1 vanish and $f = 0$.

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(III) Since $HC(W)$ is dense in W, we have $\overline{HC(W)} = W$. We still need $HC(\bar{N}) = N$ for a sub-object N of $HC(V)$. By a) this reduces to $N(\pi) = \bar{N}(\pi)$ for $\pi \in \hat{K}$. But $\bar{N}(\pi)$ is contained in $V(\pi)$, thus it is finite dimensional, and clearly $N(\pi)$ is dense in $\bar{N}(\pi)$, so we win again.

The key finiteness theorem

(I) The next theorem is fundamental.

Theorem (Harish-Chandra) If $M \in (\mathfrak{g}, K) - Mod$ is finitely generated as $U(\mathfrak{g})$-module, then $M(\pi)$ is finitely generated over $Z(\mathfrak{g})$ for any $\pi \in \hat{K}$.

We will discuss the very technical proof later on, let's focus on the many and important consequences.

The key finiteness theorem

(I) A first important consequence is

Theorem A $(\mathfrak{g}, K)$-module generated over $U(\mathfrak{g})$ by finitely many $Z(\mathfrak{g})$-finite vectors is admissible.

Say M is generated by $v_1, \ldots, v_n$, with v_i killed by some ideal J of finite codimension in $Z(\mathfrak{g})$. If $\pi \in \hat{K}$, then $M(\pi)$ is finitely generated over $Z(\mathfrak{g})$ (by the previous theorem) and killed by J, thus a finitely generated $Z(\mathfrak{g})/J$-module and a finite dimensional $\mathbb{C}$-vector space.

Irreducibility and admissibility

(I) Here is a first important application:

Theorem Any irreducible $(\mathfrak{g}, K)$-module is admissible.

Say M is irreducible, let $v \in M$ nonzero and pick a basis $v_1, \dots, v_d$ of $\mathbb{C}[K]v$. Then v_i generate M as a $U(\mathfrak{g})$-module and they are $Z(\mathfrak{g})$-finite by Dixmier's theorem. So we win thanks to the previous theorem.

Irreducibility and admissibility

(I) The analogue of the previous result fails in $\mathrm{Rep}(G(\mathbb{R}))$ (counterexamples are not easy to find!), but holds if we add a unitarity hypothesis:

Theorem Any irreducible **unitary** $G(\mathbb{R})$-representation is admissible.

Say V is irreducible unitary and let $\pi \in \hat{K}$. Let $v \in V^\infty \setminus \{0\}$. By Segal's theorem v is $Z(\mathfrak{g})$-finite. The key input is the following

Lemma Let $V \in \mathrm{Rep}(G(\mathbb{R}))$ and $v \in HC(V)$ be $Z(\mathfrak{g})$-finite. Then $M = U(\mathfrak{g})\mathbb{C}[K]v$ is admissible, its closure $\bar{M}$ is the closure of $\mathbb{C}[G(\mathbb{R})]v$ and $\bar{M}(\pi) = M(\pi)$ for $\pi \in \hat{K}$.

Irreducibility and admissibility

(I) By the lemma the closure of $M = U(\mathfrak{g})\mathbb{C}[K]v$ is V (by irreducibility of V) and $V(\pi) = \bar{M}(\pi) = M(\pi)$ is finite dimensional, so V is admissible.

Irreducibility and admissibility

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(II) Let us prove the lemma. Let W be the closure of $\mathbb{C}[G(\mathbb{R})]v$. Clearly $M \subset W$, thus $\bar{M} \subset W$. If the inclusion is strict, by Hahn-Banach there is $l \in W^*$ nonzero vanishing on M. The derivatives of the real analytic function $g \to l(gv)$ vanish at 1 and we easily get a contradiction.

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(III) Next, by a previous theorem M is admissible. Since $M(\pi)$ is dense in $\bar{M}(\pi)$ and $M(\pi)$ is finite dimensional, we have $M(\pi) = \bar{M}(\pi)$, finishing the proof.

The harmonicity theorem

(I) Finally, we can also prove the harmonicity theorem:

Theorem (Harish-Chandra)
Let $V \in \mathrm{Rep}(G(\mathbb{R}))$ and let $v \in HC(V)$ be a $Z(\mathfrak{g})$-finite vector. There is $f \in C_c^\infty(G(\mathbb{R}))$, invariant by conjugation by K and such that $v = f.v$.

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(II) Let J be the space of functions $f \in C_c^\infty(G(\mathbb{R}))$, invariant under conjugation by K. It contains a Dirac sequence, thus v is in the closure of $J.v$, thus it suffices to prove that $J.v$ is finite dimensional.

The harmonicity theorem

(I) Let $M = U(\mathfrak{g})\mathbb{C}[K]v$. By the above lemma, $\bar{M}$ is $G(\mathbb{R})$-stable, thus also J-stable, and moreover $M = \bigoplus_{\pi \in \hat{K}} \bar{M}(\pi)$, with each $\bar{M}(\pi) = M(\pi)$ finite dimensional.

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(II) Since elements of J are invariant under conjugation by K, they preserve each $\bar{M}(\pi)$. Now $v \in M$, thus there are finitely many π_i such that $v \in \sum_i M(\pi_i)$ and by the previous discussion $J.v \subset \sum_i M(\pi_i)$ is finite dimensional, finishing the proof.

Proof of the finiteness theorem

(I) Recall that we want to prove

Theorem (Harish-Chandra) If $M \in (\mathfrak{g}, K) - \mathit{Mod}$ is finitely generated as $U(\mathfrak{g})$-module, then $M(\pi)$ is finitely generated over $Z(\mathfrak{g})$ for any $\pi \in \hat{K}$.

This needs a lot of preparation...

Filtration on $U(\mathfrak{g})$

(I) Let $U_0 = \mathbb{C}$ and for $n \geq 1$ let

$$U_n = \mathrm{Span}_{X_1,\ldots,X_k \in \mathfrak{g}, k \leq n} X_1 \ldots X_k.$$

The U_n form an increasing sequence of finite dimensional $\mathbb{C}$-vector spaces with union $U(\mathfrak{g})$ and $U_n U_m \subset U_{n+m}$. This induces a filtration on $U(\mathfrak{g})$ and

$$\mathrm{gr}(U(\mathfrak{g})) = U_0 \oplus U_1/U_0 \oplus U_2/U_1 \oplus \ldots$$

is naturally a $\mathbb{C}$-algebra. A simple exercise shows that this algebra is commutative, so the natural map

$$\mathfrak{g} \to U(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$$

extends to a map of $\mathbb{C}$-algebras

$$S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g})),$$

which can be shown (exercise) to be an isomorphism.

Study of the center

(I) Let's consider now the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. By definition

$$Z(\mathfrak{g}) = \{D \in U(\mathfrak{g}) |\ DX = XD, \forall X \in \mathfrak{g}\}$$

is the centralizer of $\mathfrak{g}$. The adjoint action of G on $\mathfrak{g}$ extends to an action on $U(\mathfrak{g})$, preserving each U_n and making $U(\mathfrak{g})$ an algebraic representation of G. Since G is connected, one easily checks that

$$Z(\mathfrak{g}) = U(\mathfrak{g})^G$$

and since G is reductive (thus passage to G-invariants is exact on algebraic representations) we obtain

$$\mathrm{gr}(Z(\mathfrak{g})) = \mathrm{gr}(U(\mathfrak{g})^G) = \mathrm{gr}(U(\mathfrak{g}))^G \simeq S(\mathfrak{g})^G,$$

for the natural filtration on $Z(\mathfrak{g})$ induced by $U(\mathfrak{g})$.

Chevalley's theorem

(I) The algebra $S(\mathfrak{g})^G = S(\mathfrak{g})^{\mathfrak{g}}$ was described by Chevalley and the result is stunningly beautiful: it is a polynomial algebra in r variables, where r is the dimension of a maximal torus T in G. More precisely, let $W = N_G(T)/T$ be the Weyl group of the pair (G, T).

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(II) There is a G-equivariant isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ (pick an embedding $G \subset \mathbb{GL}_n(\mathbb{C})$ and use the G-invariant bilinear form $(X, Y) \to \mathrm{Tr}(XY)$ on $\mathfrak{g}$), so we can identify

$$S(\mathfrak{g}) \simeq S(\mathfrak{g}^*) \simeq \mathbb{C}[\mathfrak{g}]$$

in a G-equivariant way, thus $S(\mathfrak{g})^G$ is isomorphic to the ring of polynomial functions on $\mathfrak{g}$ invariant under the adjoint action of G.

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(I) There is a natural restriction map

$$\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{t}]^W,$$

where $T = \mathrm{Lie}(T)$ and Chevalley's famous theorem is

Theorem (Chevalley's restriction theorem) The above map is an isomorphism and $\mathbb{C}[\mathfrak{t}]^W$ is a polynomial algebra in $\dim T$ generators.

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(II) The proof requires a delicate study of the finite dimensional representations of G (there are ways to avoid it, though, but still the argument is intricate), but the case $G = \mathbb{GL}_n(\mathbb{C})$ is an excellent exercise!

Back to our business

(I) We are finally in good shape for the proof of the theorem. Pick generators $m_1, \ldots, m_n$ of M over $U(\mathfrak{g})$ and set $V = \sum \mathbb{C}[K] m_i$, then the obvious map $U(\mathfrak{g}) \otimes_{\mathbb{C}} V \to M$ descends to a surjection

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(II) It suffices to prove that $\mathrm{Hom}_K(\pi, U(\mathfrak{g}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} V)$ is finitely generated over $Z(\mathfrak{g})$. Let

$$W = V \otimes_{\mathbb{C}} \pi^*, \ N = U(\mathfrak{g}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} W,$$

then we need to show that N^K is finitely generated over $Z(\mathfrak{g})$.

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(I) The PBW filtration on $U(\mathfrak{g})$ induces one on N, preserved by the action of K, and a simple argument shows that it suffices to prove that $\mathrm{gr}(N^K)$ is finitely generated over $\mathrm{gr}(Z(\mathfrak{g}))$. Since K is compact, we have $\mathrm{gr}(N^K) \simeq (\mathrm{gr}(N))^K$.

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(III) Thus it suffices to prove that $(S(\mathfrak{g})/\mathfrak{k}_{\mathbb{C}} S(\mathfrak{g}) \otimes_{\mathbb{C}} W)^K$ is finitely generated over $\mathrm{gr}(Z(\mathfrak{g}))$.

Back to our business

(I) By the Cartan-Chevalley-Mostow theorem WLOG $G(\mathbb{R})$ is self-adjoint, i.e. stable under transpose, and

$$K = G(\mathbb{R}) \cap U(n).$$

The Cartan involution $\theta : G(\mathbb{R}) \to G(\mathbb{R}), g \to (g^T)^{-1}$ induces a decomposition

$$\mathfrak{g}_{\mathbb{R}} := \mathrm{Lie}(G(\mathbb{R})) = \mathfrak{k} \oplus \mathfrak{p},$$

$$\mathfrak{k} = \mathfrak{g}_{\mathbb{R}}^{\theta=1},\ \mathfrak{p} = \mathfrak{g}_{\mathbb{R}}^{\theta=-1}.$$

Back to our business

(I) The decomposition $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$ induces an isomorphism

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(III) Let $\mathfrak{a}$ be a maximal commutative subspace of $\mathfrak{p}$.

Back to our business

(I) We need the following tricky result (easy for $\mathbb{GL}_n$):

Theorem We have $\mathfrak{p} = \cup_{k \in K} \mathrm{Ad}(k)(\mathfrak{a})$.

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Theorem We have $\mathfrak{p} = \cup_{k \in K} \mathrm{Ad}(k)(\mathfrak{a})$.

(II) Keep identifying elements of the symmetric algebra of $\mathfrak{g}, \mathfrak{p}_{\mathbb{C}}, \ldots$ with polynomial functions on $\mathfrak{g}, \mathfrak{p}_{\mathbb{C}}, \ldots$. The theorem implies that that restriction to $\mathfrak{a}_{\mathbb{C}}$ induces an embedding

$$(S(\mathfrak{p}_{\mathbb{C}}) \otimes_{\mathbb{C}} W)^K \subset \mathbb{C}[\mathfrak{a}_{\mathbb{C}}] \otimes_{\mathbb{C}} W,$$

so (since $\mathbb{C}[\mathfrak{g}]^G$ is noetherian) it suffices to prove that the restriction map $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{a}_{\mathbb{C}}]$ is finite.

Back to our business

(I) But one can check that $\mathfrak{a}_{\mathbb{C}}$ is the Lie algebra of a maximal torus in G, so the result follows from Chevalley's restriction theorem.

Harish-Chandra's isomorphism

(I) Harish-Chandra used the previous theorem to prove his famous theorem describing $Z(\mathfrak{g})$. To state it, pick a Borel subgroup B containing T and let N be its unipotent radical. Let $\mathfrak{n} = \mathrm{Lie}(N)$ and $\mathfrak{b} = \mathrm{Lie}(B)$ and consider

$$M = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n} \simeq U(\mathfrak{g}).$$

There is a natural embedding $U(\mathfrak{t}) \subset M$ and $U(\mathfrak{t}) \simeq S(\mathfrak{t})$ since T is commutative. The proof of the next result is not very hard:

Theorem For any $a \in Z(\mathfrak{g})$ there is a unique $x \in U(\mathfrak{t})$ such that the image of a in M is the same as the image of x. Sending a to x yields a homomorphism of algebras

$$\varphi : Z(\mathfrak{g}) \to U(\mathfrak{t}).$$

Harish-Chandra's isomorphism

(I) Let $\rho \in \frac{1}{2}X(T)$ be half the sum of the positive roots attached to (G, B, T), i.e. the roots appearing in $\mathfrak{n}$. We define a new action of W on $\mathfrak{t}^*$ by

$$w_\bullet \lambda = w(\lambda + \rho) - \rho.$$

This induces an action of W on $S(\mathfrak{t}) \simeq \mathbb{C}[\mathfrak{t}^*]$.

Theorem (Harish-Chandra's isomorphism) The map $Z(\mathfrak{g}) \to S(\mathfrak{t})$ in the previous theorem induces an isomorphism

$$Z(\mathfrak{g}) \simeq S(\mathfrak{t})^W$$

and this is a polynomial algebra in $\dim T$ generators.

Harish-Chandra's isomorphism

(I) The hard part in the proof is showing that the image of φ is invariant under W, which is done by some explicit computations with **Verma modules**, i.e. quotients of the form $M_\lambda = M \otimes_{U(\mathfrak{t})} \mathbb{C}$ for $\lambda : \mathfrak{t} \to \mathbb{C}$. Once this is achieved, one checks without much pain that φ induces on the associated graded rings precisely Chevalley's restriction isomorphism.

The proof of the finiteness theorem: the finale

(I) Let now G be a connected reductive group over $\mathbb{Q}$ and let Γ be an arithmetic subgroup of $G(\mathbb{Q})$. We want to prove that for any ideal J of finite codimension in $Z(\mathfrak{g})$ and any $\pi_1, \ldots, \pi_n \in \hat{K}$ the space of $f \in \mathscr{A}(G, \Gamma)$ of types J and $\pi_1, \ldots, \pi_n$ is finite dimensional. We proved this last time for the cuspidal subspace, and also explained a reduction to the case $A_G = 1$ (A_G being the split component of G).

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(II) To prove the result in general we induct on the $\mathbb{Q}$-rank of G, i.e. the dimension of the maximal $\mathbb{Q}$-split tori in G. If this is 0, then G is anisotropic, so all forms are cuspidal and we are done. Say this is > 0. If there are no proper $\mathbb{Q}$-parabolics in G we are done by the same argument, so suppose that this is not the case. We saw last time that the set of $\mathbb{Q}$-parabolics up to Γ-conjugacy is finite, pick representatives $P_1, \ldots, P_r$.

The proof of the finiteness theorem: the finale

(I) Let $f \in \mathscr{A}(G, \Gamma)$ and consider $f_i = f_{P_i}$, the constant term along each P_i. By properties of the constant term, the kernel of the map $\varphi : f \to (f_{P_1}, ..., f_{P_r})$ consists of cusp forms, so the restriction of the kernel to forms of type $J, \pi_1, ..., \pi_n$ is finite dimensional (the main result of the last lecture). So it suffices to prove that the image of $\mathscr{A}(G, \Gamma)[J, \pi_1, ..., \pi_r]$ is finite dimensional. Let $L_i = N_i / P_i$ be the Levi quotient of P_i, with N_i the unipotent radical of P_i. We will see below that f_i are automorphic forms on L_i for the arithmetic subgroups Γ_i (image of $P_i \cap \Gamma$ in L_i), with K and $Z(\mathfrak{g})$-types determined by J and the π_i. By the inductive hypothesis (the L_i have smaller $\mathbb{Q}$-rank than G) $\varphi(\mathscr{A}(G, \Gamma)[J, \pi_1, ..., \pi_r])$ is finite dimensional and so we win!

The proof of the finiteness theorem: the finale

(I) We are thus reduced to the following statement: for a proper $\mathbb{Q}$-parabolic P with unipotent radical N and Levi quotient $L = N\backslash P$, for any $f \in \mathscr{A}(G, \Gamma)[J, \pi_1, ..., \pi_r]$ the constant term f_P defines an automorphic form on L with respect to Γ_L (image of $P \cap \Gamma$) of K and $Z(\mathfrak{g})$-types specified by J and the π_i.

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(II) First, by design

$$f_P(g) = \int_{N(\mathbb{R})\cap\Gamma\backslash N(\mathbb{R})} f(ng)dn$$

is left $N(\mathbb{R})$-invariant and also left $P\cap\Gamma$-invariant, thus it defines a function on $L(\mathbb{R}) \simeq N(\mathbb{R})/P(\mathbb{R})$ which is left Γ_L-invariant, obviously smooth and of moderate growth.

The proof of the finiteness theorem: the finale

(I) Let $M_P, A_P, \ldots$ the factors in the Langlands decomposition of $P(\mathbb{R})$. Then $K \cap M_P$ is a maximal compact subgroup of $P(\mathbb{R})$ and its image K_L in $L(\mathbb{R})$ is a maximal compact subgroup of $L(\mathbb{R})$. Using this it is clear that f_P is K_L-finite, of type specified by the π_i.

The proof of the finiteness theorem: the finale

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(II) The hard part is proving that f_P is $Z(\mathfrak{l})$-finite, of type specified by J. The same argument as in the construction of the Harish-Chandra isomorphism yields a homomorphism

$$\varphi_{\mathfrak{l}} : Z(\mathfrak{g}) \to Z(\mathfrak{l})$$

such that $D - \varphi_{\mathfrak{l}}(D) \in U(\mathfrak{g})\mathfrak{n}$ for $D \in Z(\mathfrak{g})$.

The proof of the finiteness theorem: the finale

(I) Since f_P is left $N(\mathbb{R})$-invariant, it is killed by $\mathfrak{n}$ and thus $\varphi_{\mathfrak{l}}(J)Z(\mathfrak{l})$ kills f_P. It suffices to show that this ideal has finite codimension in $Z(\mathfrak{l})$ and for this it suffices to show that $\varphi_{\mathfrak{l}}$ is finite. Again, passing to graded pieces it suffices to check that $S(\mathfrak{g})^G \to S(\mathfrak{l})^L$ is finite. With the usual identification $\mathfrak{g} \simeq \mathfrak{g}^*$, this is just the restriction map. The result follows then easily from the Chevalley restriction theorem.