Lecture 10: "classical" automorphic forms on reductive groups

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What is an automorphic form?

(I) **In the sequel G will be a connected reductive $\mathbb{Q}$-group, with a fixed maximal compact subgroup K of $G(\mathbb{R})$ and a fixed arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$.**

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(II) The space of automorphic forms of level Γ on G (rel. to K)

$$\mathscr{A}(G, \Gamma) \subset C^{\infty}(\Gamma \backslash G(\mathbb{R}))$$

consists of those $f \in C^{\infty}(\Gamma \backslash G(\mathbb{R}))$ such that:

- f is right K-finite, i.e. $\dim \mathrm{Span}_{k \in K} f(\bullet k) < \infty$.
- f is $\mathfrak{Z}$-finite (see below)
- f has moderate growth (see below).

$\mathfrak{Z}$-finiteness

(I) Recall that the (complex) Lie algebra $\mathfrak{g}$ of G acts on $C^\infty(G(\mathbb{R}))$ by

$$X.f(g) = \lim_{t \to 0} \frac{f(g \exp(tX)) - f(g)}{t}.$$

Let $U(\mathfrak{g})$ (**enveloping algebra** of $\mathfrak{g}$) be the sub-algebra of $\mathrm{End}_{\mathbb{C}}(C^\infty(G(\mathbb{R})))$ generated by $f \to X.f$ for $X \in \mathfrak{g}$.

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(II) It follows from theorems of Chevalley and Harish-Chandra (cf. next lectures) that the center

$$\mathfrak{Z}(\mathfrak{g}) = Z(U(\mathfrak{g}))$$

of $U(\mathfrak{g})$ is a polynomial algebra in as many generators as the rank of a maximal torus in G.

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of $U(\mathfrak{g})$ is a polynomial algebra in as many generators as the rank of a maximal torus in G.

(III) A map $f \in C^\infty(G(\mathbb{R}))$ is called **$\mathfrak{Z}$-finite** if

$$\dim \mathrm{Span}_{X\in\mathfrak{Z}(\mathfrak{g})} X.f < \infty.$$

Moderate growth

(I) Pick a $\mathbb{Q}$-embedding $G \subset \mathbb{GL}_n(\mathbb{C})$. We have a natural norm on $\mathbb{GL}_n(\mathbb{R})$, inducing one on $G(\mathbb{R})$

$$||g|| = \sqrt{\mathrm{Tr}({}^t gg) + 1/\det(g)^2}.$$

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(II) A map $f : G(\mathbb{R}) \to \mathbb{C}$ has **moderate growth (or MG)** if there are c, N such that

$$|f(g)| \leq c||g||^N, \ \forall g \in G(\mathbb{R}).$$

This notion is independent (exercise!) of the choice of the embedding of G in $\mathbb{GL}_n(\mathbb{C})$.

Applications of harmonicity

(I) $\mathscr{A}(G,\Gamma)$ is contained in $C^\infty(\Gamma\backslash G(\mathbb{R}))$, but it is not stable under $G(\mathbb{R})$ (because of the K-finiteness condition). Remarkably, it is stable under the infinitesimal action of $G(\mathbb{R})$, i.e. under $\mathfrak{g}$. This is not trivial at all because of the MG condition, but, as for $\mathbb{SL}_2$, this follows from the harmonicity theorem (valid in this degree of generality, cf. next lectures). More precisely:

Theorem (Harish-Chandra) Any $f \in \mathscr{A}(G,\Gamma)$ is real analytic on $G(\mathbb{R})$, satisfies $f = f * \alpha$ for some $\alpha \in C_c^\infty(G(\mathbb{R}))$, and has **uniform moderate growth**: there is N such that for all $X \in U(\mathfrak{g})$ we have

$$\sup_{g\in G(\mathbb{R})} \frac{|X.f(g)|}{||g||^N} < \infty.$$

The finiteness theorem

(I) Since $\mathscr{A}(G, \Gamma)$ is clearly stable under K, it is a sub-$(\mathfrak{g}, K)$-module of $C^\infty(\Gamma\backslash G(\mathbb{R}))$.

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(II) Our goal for today is to start the (sketch of the) proof of the following deep and fundamental result, at the basis of the theory:

Theorem (Harish-Chandra's finiteness theorem) For any ideal J of finite codimension in $\mathfrak{Z}(\mathfrak{g})$ the $(\mathfrak{g},K)$-module

$$\mathscr{A}(G,\Gamma)[J] = \{f \in \mathscr{A}(G,\Gamma) |\, J.f = 0\}$$

is admissible, i.e. for any $\pi \in \hat{K}$

$$\dim \mathrm{Hom}_K(\pi, \mathscr{A}(G,\Gamma)[J]) < \infty.$$

Dévissage via the split component

(I) We say that $f \in C^\infty(\Gamma\backslash G(\mathbb{R}))$ has $\mathfrak{Z}$-**type** J if $J.f = 0$ and K-**type** $\pi_1, .., \pi_r \in \hat{K}$ if

$$\mathbb{C}[K].f \simeq \bigoplus_{i=1}^{r} \pi_i.$$

The theorem is equivalent to: for any ideal J of finite codimension in $\mathfrak{Z}(\mathfrak{g})$ and any $\pi_1, ..., \pi_r$ the space of forms $f \in \mathscr{A}(G, \Gamma)$ of types J and $\pi_1, ..., \pi_r$ is finite dimensional.

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(II) We will first reduce the theorem to the case $A_G = 1$ (introduced below), then prove it for cuspidal forms (to be defined...) and finally deduce it by a rather subtle inductive argument.

Dévissage via the split component

(I) Let Z_{spl} be the largest $\mathbb{Q}$-split torus contained in the centre of G. The **split component** A_G of G is

$$A_G = Z_{\mathrm{spl}}(\mathbb{R})^0.$$

Let $X(G)_{\mathbb{Q}}$ be the set of characters $G \to \mathbb{G}_m$ defined over $\mathbb{Q}$. The subgroup

$${}^0G = \{g \in G \mid \chi(g)^2 = 1 \ \forall \chi \in X(G)_{\mathbb{Q}}\}$$

of G is Zariski closed, defined over $\mathbb{Q}$, normal, thus reductive, though not necessarily connected.

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(II) For automorphic business all the hard part is in ${}^0G$: if G is semi-simple ${}^0G = G$ and $A_G = \{1\}$.

Dévissage via the split component

(I) The objects A_G and $X(G)_{\mathbb{Q}}$ are quite simple:

• the map $X(G)_{\mathbb{Q}} \to X(Z_{\mathrm{spl}})_{\mathbb{Q}}$ is easily seen to be injective, with finite cokernel. If $k = \mathrm{rk}(Z_{\mathrm{spl}})$, we thus have

$$X(G)_{\mathbb{Q}} \simeq \mathbb{Z}^k, \ A_G \simeq \mathbb{R}^k_{>0}.$$

• letting $\mathfrak{a}_G = \mathrm{Lie}(A_G)$, the exponential map is an isomorphism $\mathfrak{a}_G \simeq A_G$. If $\log : A_G \to \mathfrak{a}_G$ is its inverse, then $\chi \to d\chi(1)$ and $\lambda \to (a \to e^{\lambda(\log a)})$ give inverse bijections

$$X(A_G) := \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{gr}}(A_G, \mathbb{R}_{>0}) \simeq \mathfrak{a}^*_G.$$

Moreover, $X(G)_{\mathbb{Q}}$ is a lattice in $\mathfrak{a}^*_G$ via

$$X(G)_{\mathbb{Q}} \otimes \mathbb{R} \simeq X(A_G) \simeq \mathfrak{a}^*_G.$$

Dévissage via the split component

(I) One checks that $G(\mathbb{R}) =^0 G(\mathbb{R}) \times A_G$ and that ${}^0G(\mathbb{R})$ contains $G(\mathbb{R})_{\rm der}$ and any compact subgroup of $G(\mathbb{R})$. The arithmetic subgroup Γ of $G(\mathbb{Q})$ is contained in ${}^0G(\mathbb{R})$ and a lattice in there (Borel, Harish-Chandra theorem).

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(II) If $f \in C^\infty(\Gamma\backslash G(\mathbb{R}))$ is $\mathfrak{Z}$-finite, an exercise in PDE shows

$$f(x,a) = \sum_{i=1}^{d} Q_i(a)(X_i \bullet f)(x),\ (x,a) \in G(\mathbb{R}) =^0 G(\mathbb{R}) \times A_G$$

for some

$$X_i \in \mathfrak{Z}(\mathfrak{g}),\ \ Q_i \in \mathbb{C}[\mathfrak{a}_G] \otimes X(A_G).$$

Thus each Q_i is a finite sum of functions of the form $a \to e^{\lambda(\log a)} P(\log a)$ with $\lambda \in \mathfrak{a}_G^*$ and P a polynomial function on $\mathfrak{a}_G$.

Dévissage via the split component

(I) If f is automorphic of types J and $\pi_1, \ldots, \pi_r \in \hat{K}$ then $X_i \bullet f$ are automorphic for ${}^0G$, of types $\mathfrak{Z}({}^0\mathfrak{g}) \cap J$ and $\pi_1, \ldots, \pi_r$. The Q_i are killed by $J \cap U(\mathfrak{a}_G)$, of finite codimension in $U(\mathfrak{a}_G)$. An exercise in analysis shows that the Q_i live in a finite dimensional vector space, reducing the finiteness theorem to the case $A_G = 1$.

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(II) The next step is to study the cuspidal space, as we did for $\mathbb{SL}_2$. This requires the fundamental notion of parabolic subgroups. They play a key role in the theory (as it was clear for $\mathbb{SL}_2$), controlling the behavior at ∞. The theory is however much more involved for general G than for $\mathbb{SL}_2$...

Parabolic subgroups

(I) For a Zariski closed subgroup P of G the following statements are (very nontrivially) equivalent, in which case we say that P is a **parabolic subgroup** of G:

- $G(\mathbb{C})/P(\mathbb{C})$ is a compact topological space.

- P contains a Borel subgroup of G.

- there is a morphism of algebraic groups $\lambda : \mathbb{G}_m \to G$ such that

$$P = P(\lambda) := \{g \in G | \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists in } \mathbb{C}\}.$$

Parabolic subgroups

(I) If $G = \mathbb{GL}_n(\mathbb{C})$, the last description implies that parabolic subgroups are the stabilisers of flags

$$\{0\} = V_0 \subset V_1 \subset ... \subset V_s = \mathbb{C}^n,$$

i.e. block upper triangular matrices in which the diagonal blocks have sizes $n_1, ..., n_s$, with $n_i = \dim V_i / V_{i-1}$ satisfying $n = n_1 + ... + n_s$.

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(II) G has no proper (i.e. different from G) $\mathbb{Q}$-parabolic if and only if G_{der} is $\mathbb{Q}$-anisotropic, in which case the automorphic life is not hard: if moreover $A_G = 1$ (i.e. G is $\mathbb{Q}$-anisotropic), then all automorphic forms are bounded since $\Gamma \backslash G(\mathbb{R})$ is compact, and the finiteness theorem is a (relatively) simple consequence of Godement's lemma.

Constant term, cusp forms

(I) Let P be a $\mathbb{Q}$-parabolic of G, with unipotent radical N. Then $N(\mathbb{R}) \cap \Gamma$ is a co-compact lattice in $N(\mathbb{R})$ (exercise!) and we get a map, **constant term along** P

$$C(\Gamma\backslash G(\mathbb{R})) \to C(N(\mathbb{R})\backslash G(\mathbb{R})), f \to f_P,$$

$$f_P(g) = \int_{N(\mathbb{R})\cap\Gamma\backslash N(\mathbb{R})} f(ng)dn, \;\; g \in G(\mathbb{R}),$$

where dn is the Haar measure on $N(\mathbb{R})$ giving $N(\mathbb{R}) \cap \Gamma\backslash N(\mathbb{R})$ mass 1.

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(II) Say f is **cuspidal or cusp form** if $f_P = 0$ for any **proper** $\mathbb{Q}$-parabolic P. If G_{der} is $\mathbb{Q}$-anisotropic, then any f is trivially cuspidal!

Constant term, cusp forms

(I) Simple exercises show that if $f_P = 0$ for a proper $\mathbb{Q}$-parabolic P, then $f_Q = 0$ for any $\mathbb{Q}$-parabolic $Q \subset P$. Also, for $\gamma \in \Gamma$ we have $f_{\gamma^{-1} P \gamma}(g) = f_P(\gamma g)$.

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(II) The next deep theorem shows that there are only finitely many such vanishing conditions to check.

Theorem (Borel, Harish-Chandra) There are only finitely many $\mathbb{Q}$-parabolics of G up to Γ-conjugacy.

This has several serious inputs: a theorem of Borel-Tits ensuring that up to $G(\mathbb{Q})$-conjugacy there are only finitely many $\mathbb{Q}$-parabolics in G, a theorem of Chevalley ensuring that the normaliser of a parabolic is the parabolic itself, and reduction theory (Borel-HC) which ensures that for any $\mathbb{Q}$-parabolic P the set $\Gamma\backslash G(\mathbb{Q})/P(\mathbb{Q})$ is finite (this set classifies the $G(\mathbb{Q})$-conjugates of P up to Γ-conjugacy).

The GPS theorem

(I) We can now state two fundamental theorems. **We assume that $A_G = 1$ for both, so $\mathrm{vol}(\Gamma\backslash G(\mathbb{R})) < \infty$.**

Theorem (Gelfand, Piatetski-Shapiro) a) Any $f \in \mathscr{A}(G,\Gamma)_{\mathrm{cusp}}$ is bounded, thus

$$\mathscr{A}(G,\Gamma)_{\mathrm{cusp}} \subset L^2(\Gamma\backslash G(\mathbb{R}))_{\mathrm{cusp}}.$$

b) For any $\alpha \in C_c^\infty(G(\mathbb{R}))$ there is $c > 0$ such that

$$||f * \alpha||_\infty \leq c||f||_{L^2}, \ \forall f \in L^2(\Gamma\backslash G(\mathbb{R}))_{\mathrm{cusp}}.$$

The operator $f \to f * \alpha$ on $L^2(\Gamma\backslash G(\mathbb{R}))_{\mathrm{cusp}}$ is Hilbert-Schmidt and $L^2(\Gamma\backslash G(\mathbb{R}))_{\mathrm{cusp}}$ has a discrete decomposition.

The weak finiteness theorem

(I) See the end of the lecture for a sketch of the very technical proof. We deduce now a weak form of the finiteness theorem: the space

$$X = \mathscr{A}(G, \Gamma)_{\mathrm{cusp}}[J, \pi_1, \ldots, \pi_r]$$

of **cusp forms** of $\mathfrak{Z}$-type J and K-type $\pi_1, \ldots, \pi_r$ is finite dimensional. By Godement's lemma and the GPS theorem above it suffices to show that X is closed in $L^2(\Gamma \backslash G(\mathbb{R}))$, which, as for $\mathbb{SL}_2$, is highly nontrivial. Say $f_n \in X$ tend to $f \in L^2(\Gamma \backslash G(\mathbb{R}))$. Simple applications of Cauchy-Schwarz show that f is of K-type $\pi_1, \ldots, \pi_r$.

The weak finiteness theorem

(I) Next, as for $\mathbb{SL}_2$ we interpret f as a distribution on $G(\mathbb{R})$ and we show that this distribution is killed by $\mathfrak{Z}(\mathfrak{g})$. The key point is that $U(\mathfrak{g})$ has an anti-automorphism $D \to \check{D}$ such that $\check{D} = -D$ for $D \in \mathfrak{g}$ and for $f \in C^\infty(G(\mathbb{R}))$ and $\varphi \in C_c^\infty(G(\mathbb{R}))$

$$\int_{G(\mathbb{R})} (D.f)(g)\varphi(g)dg = \int_{G(\mathbb{R})} (\check{D}.\varphi)(g)f(g)dg.$$

We win by Cauchy-Schwarz: $f \to \int_{G(\mathbb{R})} \check{D}.\varphi(g)f(g)dg$ is continuous for the L^2 norm as $\check{D}.\varphi \in C_c(G(\mathbb{R}))$.

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(II) Now the distribution f is $\mathfrak{Z}$-finite and K-finite, thus by elliptic regularity f is real analytic (up to a set of measure 0) and killed by J.

The weak finiteness theorem

(I) At this moment we can invoke harmonicity to get the existence of $\alpha \in C_c^\infty(G(\mathbb{R}))$ with $f = f * \alpha$. Since Γ is a lattice in $G(\mathbb{R})$, we have $f \in L^1(\Gamma \backslash G(\mathbb{R}))$ and we get that f has moderate growth via the next theorem, whose proof identical to the case $\mathbb{SL}_2$ (the subtle counting lemma used there is actually useless since Γ is arithmetic):

Theorem (first fundamental estimate) There is N such that for any $\alpha \in C_c^\infty(G)$

$$\sup_{f \in L^1(\Gamma \backslash G(\mathbb{R})) \backslash \{0\}, x \in G(\mathbb{R})} \frac{|(f * \alpha)(x)|}{||x||^N} < \infty.$$

The weak finiteness theorem

(I) Finally, we need to prove that f is cuspidal. Pick a $\mathbb{Q}$-parabolic P, with unipotent radical N. Since f_P is left $N(\mathbb{R})$-invariant, it suffices to check that for any $\varphi \in C_c(N(\mathbb{R})\backslash G(\mathbb{R}))$ we have

$$\int_{N(\mathbb{R})\backslash G(\mathbb{R})} \varphi(g) f_P(g) = 0.$$

Unfolding (using that f is left Γ-invariant) gives

$$\int_{N(\mathbb{R})\backslash G(\mathbb{R})} \varphi(g) f_P(g) = \int_{N(\mathbb{R})\cap\Gamma\backslash G(\mathbb{R})} \varphi(g) f(g) dg$$

$$= \int_{\Gamma\backslash G(\mathbb{R})} f(g) H(g) dg, \text{where } H(g) = \sum_{\gamma\in N(\mathbb{R})\cap\Gamma\backslash\Gamma} \varphi(\gamma g).$$

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(II) But one checks that H is bounded (easy exercise), so $f \to \int_{\Gamma\backslash G(\mathbb{R})} f(g)H(g)dg$ is continuous for the L^2 norm and $f_P = 0$.

Iwasawa and Langlands decompositions

(I) In order to properly discuss the GPS theorem we need serious and quite technical background. Let P be any $\mathbb{Q}$-parabolic of G, with unipotent radical N. Then the **Levi quotient** $L_P = P/N$ of P is a connected reductive $\mathbb{Q}$-group (P is connected by a fundamental theorem of Chevalley).

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(II) One can find a $\mathbb{Q}$-**Levi subgroup** L of P, i.e. such that $L \to P \to L_P$ is an isomorphism (equivalently $LN = P$ is a semi-direct product). Indeed, there is $\lambda \in \mathrm{Hom}(\mathbb{G}_m, G)_{\mathbb{Q}}$ such that $P = P(\lambda)$, and then one checks that the centraliser of the image of λ is a $\mathbb{Q}$-Levi subgroup.

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(III) Pick such a $\mathbb{Q}$-Levi subgroup $L \subset P$. Since $P = LN$ is a semi-direct product, we have

$$P(\mathbb{R}) = L(\mathbb{R})N(\mathbb{R}) = N(\mathbb{R})L(\mathbb{R}),\ L(\mathbb{R}) \simeq L_P(\mathbb{R}) \simeq^0 L_P(\mathbb{R}) \times A_{L_P}.$$

Iwasawa and Langlands decompositions

(I) The maximal compact K of $G(\mathbb{R})$ is the fixed-point subgroup of a Cartan involution θ of $G(\mathbb{R})$. One shows that there are unique subgroups A_P, M_P of $P(\mathbb{R})$ which are conjugates of $A_{L_P}, {}^0L_P(\mathbb{R})$ and which are θ-stable.

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(II) We obtain the **Langlands decomposition of P**

$$P(\mathbb{R}) = N(\mathbb{R})A_P M_P = N(\mathbb{R})M_P A_P$$

and the **Iwasawa decomposition**

$$G(\mathbb{R}) = P(\mathbb{R})K = N(\mathbb{R})M_P A_P K.$$

In the last decomposition the A_P-component of $g \in G(\mathbb{R})$ is uniquely determined and denoted $a(g) \in A_P$.

Relative root system

(I) **From now on we fix a minimal $\mathbb{Q}$-parabolic P in G, with unipotent radical N.** By Borel-Tits, all such P are conjugate under $G(\mathbb{Q})$, and the set of $\mathbb{Q}$-parabolics containing P is both finite and a set of representatives for the $G(\mathbb{Q})$-conjugacy classes of $\mathbb{Q}$-parabolics of G.

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(II) Let S be a maximal $\mathbb{Q}$-split torus of G contained in P and let

$$A = S(\mathbb{R})^0, \ \mathfrak{a} = \mathrm{Lie}(A), X(A) := \mathrm{Hom}_{\mathrm{gr}}^{\mathrm{cont}}(A, \mathbb{R}_{>0}) \simeq \mathfrak{a}^*.$$

Don't confuse these with $A_G, X(A_G), \mathfrak{a}_G$!

Relative root system

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(III) Then $L = Z_G(S)$ is a $\mathbb{Q}$-Levi subgroup of P and the adjoint representation of S in $\mathfrak{g}$ gives rise to a a decomposition

$$\mathfrak{g} = \mathrm{Lie}(L) \oplus \bigoplus_{a\in\Phi} \mathfrak{g}_a$$

for some finite set $\Phi \in X(S) \setminus \{0\} \subset X(A) \setminus \{0\}$.

Relative root system

(I) The set Φ is called the **relative root system of G with respect to S** because of the deep:

Theorem (Borel-Tits) Φ is a root system in the vector space $X(A/A_G) \subset X(A)$, and there is a system of positive roots $\Phi^+ \subset \Phi$ such that

$$\mathfrak{n} := \mathrm{Lie}(N) = \sum_{a \in \Phi^+} \mathfrak{g}_a.$$

Caution: contrary to the "absolute" theory, the $\mathfrak{g}_a$ are not necessarily 1-dimensional!

Siegel sets and reduction theory

(I) Recall the Langlands decomposition $P(\mathbb{R}) = N(\mathbb{R})M_P A_P$, with A_P a suitable conjugate of A, stable under the Cartan involution of G attached to K. For $t > 0$ let

$$A_{P,t} = \{a \in A_P |\ \alpha(a) \geq t\ \forall \alpha \in \Phi^+\}.$$

If $\Delta = \{\alpha_1, ..., \alpha_l\}$ is the basis of Φ^+, it is equivalent to ask that $\alpha_i(a) \geq t$ for all i. If $A_G = 1$, the α_i form a basis of $\mathfrak{a}^*$.

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(II) A **Siegel set at** P is a set of the form

$$\Sigma = \omega A_{P,t} K$$

with $t > 0$ and $\omega \subset N(\mathbb{R})M_P$ a compact set.

Siegel sets and reduction theory

(I) The next result is a vast (and very hard) generalisation of reduction theory for $\mathbb{SL}_n$, seen last time:

Theorem (Borel, Harish-Chandra) There is a Siegel set Σ at P and a finite set $C \subset G(\mathbb{Q})$ such that $\Omega = C\Sigma$ is a good approximation of $\Gamma \backslash G(\mathbb{R})$: $G(\mathbb{R}) = \Gamma\Omega$ and there are only finitely many $\gamma \in \Gamma$ such that $\gamma\Omega \cap \Omega \neq \emptyset$.

Upshot: $\Gamma \backslash G(\mathbb{R})$ is covered by finitely many Siegel sets Σ_i, at representatives for the Γ-conjugacy classes of minimal $\mathbb{Q}$-parabolics of G.

Growth on Siegel sets

(I) **From now on we assume that $A_G = 1$ and we fix Siegel sets Σ_i at various minimal parabolics P_i, covering** $\Gamma\backslash G(\mathbb{R})$. Recall that the A_P-component $a(g)$ of any $g \in G(\mathbb{R}) = N(\mathbb{R})M_P A_P K$ is well-defined.

Theorem (Harish-Chandra) A map $f \in C(\Gamma\backslash G(\mathbb{R})$ has moderate growth if and only if there are $\lambda_i \in X(A_{P_i})$ and $c_i > 0$ such that

$$|f(x)| \leq c_i \lambda_i(a(x)), \; x \in \Sigma_i.$$

Growth on Siegel sets

(I) To prove this one needs to establish three things ($P = P_i$ for some i):

• for any $\lambda \in X(A_P)$ there are c, N such that $\lambda(a) \leq c||a||^N$ for $a \in A_{P,t}$.

• there are $c > 0$ and $\lambda \in X(A_P)$ such that $||a|| \leq c\lambda(a)$ for $a \in A_{P,t}$.

• there is $c > 0$ such that for all $g \in \Sigma_i$ and $\gamma \in \Gamma$

$$||\gamma g|| \geq c||g||.$$

The first two are relatively easy exercises, the last one is not easy (but it's a great exercise for $\mathbb{GL}_n$!).

Growth on Siegel sets

(I) For simplicity let's assume that $C = \{1\}$, i.e. we have one Siegel set Σ at our minimal $\mathbb{Q}$-parabolic P covering $\Gamma\backslash G(\mathbb{R})$. For $\lambda \in X(A_P)$ let $\Gamma_\infty = \Gamma \cap N(\mathbb{R})$ and

$$||f||_\lambda = \sup_{x\in\Sigma} |f(x)|\lambda(x),$$

$$C^\infty(\lambda) = \{f \in C^\infty(\Gamma_\infty\backslash G(\mathbb{R}))|\, ||D.f||_\lambda < \infty\ \forall D \in U(\mathfrak{g})\}.$$

Endow $C^\infty(\lambda)$ with the semi-norms $f \to ||D.f||_\lambda$ for $D \in U(\mathfrak{g})$.

By the above theorem, any automorphic form is in some $C^\infty(\lambda)$, since it has **uniform** moderate growth.

Second fundamental estimate

(I) The following theorem, really the heart of the story, allows one to "travel" between various $C^\infty(\lambda)$ using constant terms along maximal $\mathbb{Q}$-parabolic subgroups of G. It is a vast generalisation of the second fundamental estimate for $\mathbb{SL}_2$.

Second fundamental estimate

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(II) The maximal $\mathbb{Q}$-parabolics $P_1, ..., P_l$ containing P are indexed by elements of $\Delta = \{\alpha_1, ..., \alpha_l\}$ and we have $P_i = N_i Z_G(S)$, with N_i normal in N, more precisely,

$$N_i = \exp(\mathfrak{n}_i), \ \mathfrak{n}_i = \sum_{\beta \in \Phi^+ \setminus \mathrm{Span}(\Delta \setminus \{\alpha_i\})} \mathfrak{g}_\beta .$$

Second fundamental estimate

(I) Consider the operator

$$\pi_i : C^\infty(\Gamma_\infty \backslash G(\mathbb{R})) \to C^\infty(\Gamma_\infty \backslash G(\mathbb{R})), \pi_i(f) = f_{P_i}.$$

Theorem (Harish-Chandra) For $\lambda \in X(A_P), \lambda' \in \lambda + \mathbb{R}\alpha_i$, $f \to f - \pi_i(f)$ induces a continuous operator

$$1 - \pi_i : C^\infty(\lambda) \to C^\infty(\lambda').$$

The proof is quite similar to the case $\mathbb{SL}_2$, the difficulty being that $N_i(\mathbb{R})$ is not always abelian. But we can filter $N_i(\mathbb{R})$ by subgroups N_i^j normalised by A_P, with successive quotients $N_i^{j-1} \backslash N_i^j \simeq \mathbb{R}$ and A_P acts on these quotients by characters β_j such that $\beta_j(a) \geq c_j \alpha_i(a)$ for $a \in A_{P,t}$ (for some $c_i > 0$). Also, $\Gamma_\infty \cap N_i^j$ is a co-compact lattice in N_i^j. We are then reduced to Fourier analysis on $\mathbb{Z}\backslash\mathbb{R}$, as for $\mathbb{SL}_2$.

Second fundamental estimate

(I) Iterating, we deduce that $\prod_{i=1}^{l}(1-\pi_i) : C^\infty(\lambda) \to C^\infty(\lambda')$ is a continuous operator for any λ and λ' (since our assumption that $A_G = \{1\}$ ensures that the simple roots α_i span $X(A_P) \simeq \mathfrak{a}^*$).

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(II) **But** this operator is simply the identity on cusp forms, thus

$$\mathscr{A}(G,\Gamma)_{\mathrm{cusp}} \subset C^\infty(\lambda')$$

for any $\lambda' \in X(A_P)$, i.e. cusp forms are rapidly decreasing, in particular bounded on the Siegel set Σ and thus bounded on $G(\mathbb{R}) = \Gamma\Sigma$ (recall that we're assuming $C = \{1\}$). This proves the first part of the GPS theorem.

Second part of GPS

(I) Let's prove now:

Theorem (Gelfand, Piatetski-Shapiro) For any $\alpha \in C_c^\infty(G(\mathbb{R}))$ there is $c > 0$ such that

$$||f * \alpha||_\infty \leq c||f||_{L^2}, \ \forall f \in L^2_{\text{cusp}}(\Gamma \backslash G(\mathbb{R})).$$

The operator $f \to f * \alpha$ on $L^2_{\text{cusp}}(\Gamma \backslash G(\mathbb{R}))$ is Hilbert-Schmidt.

The first part implies the second one by the same (slightly subtle due to the issue of the measurability of the kernel) argument as for $\mathbb{SL}_2$: it implies that $f \to f * \alpha$ is a kernel operator, the kernel being square integrable.

Second part of GPS

(I) Set $\varphi = f * \alpha$, then for any proper $\mathbb{Q}$-parabolic P we have

$$\varphi_P = f_P * \alpha = 0$$

since f is cuspidal. Thus φ is cuspidal as well. Moreover, by the first fundamental estimate we have for all $D \in U(\mathfrak{g})$ (and a suitable N, independent of D and f)

$$|D\varphi(x)| = |f * (D.\alpha)(x)| \leq c_D ||x||^N ||f||_{L^1} \leq c'_D ||x||^N ||f||_{L^2},$$

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the last one by Cauchy-Schwarz (introducing an absolute constant depending on Γ, G).

(II) Thus there is $\lambda \in X(A_P)$ such that $\varphi \in C^\infty(\lambda)$. Since φ is cuspidal and the inclusion $C^\infty(\lambda)_{\mathrm{cusp}} \to C^\infty(0)$ is continuous, we immediately get the result thanks to the above estimate.

Second part of GPS

(I) At this point we "proved" the finiteness theorem for the cuspidal part. Next time we'll bootstrap this to the whole automorphic space, by a rather subtle inductive argument.