Lecture 10: lattices, encore et encore!

Gabriel Dospinescu

CNRS, ENS Lyon

Goal

(I) In this lecture we will explain a series of deep relations between unimodular lattices in euclidean spaces, modular forms and adelic groups. Absolutely beautiful references are the book of Serre "A Course in Arithmetic", and (much more advanced) the book of Chenevier-Lannes "Automorphic forms and even unimodular lattices".

Unimodular lattices

(I) A **unimodular quadratic lattice** of rank n is a free $\mathbb{Z}$-module L of rank n together with a symmetric bilinear pairing $L^2 \to \mathbb{Z}, (x, y) \to x \bullet y$ which is perfect, i.e. the induced map

$$L \to \mathrm{Hom}(L, \mathbb{Z})$$

is bijective. In terms of matrices, if

$$A = (e_i \bullet e_j)_{1 \le i,j \le n}$$

is the Gram matrix associated to a basis $e_1, ..., e_n$ of L, then perfectness is equivalent to $\det A \in \{-1, 1\}$.

Unimodular lattices

(I) A **unimodular quadratic lattice** of rank n is a free $\mathbb{Z}$-module L of rank n together with a symmetric bilinear pairing $L^2 \to \mathbb{Z}, (x, y) \to x{\bullet}y$ which is perfect, i.e. the induced map

$$L \to \mathrm{Hom}(L, \mathbb{Z})$$

is bijective. In terms of matrices, if

$$A = (e_i{\bullet}e_j)_{1 \leq i,j \leq n}$$

is the Gram matrix associated to a basis $e_1, ..., e_n$ of L, then perfectness is equivalent to $\det A \in \{-1, 1\}$.

(II) There is an obvious notion of isomorphism between quadratic lattices. In terms of matrices this means replacing A by ${}^TBAB$ for some $B \in \mathbb{GL}_n(\mathbb{Z})$.

Nice lattices

(I) Let $\mathscr{L}_n$ be the set of **nice** lattices, i.e. unimodular quadratic lattices (L, q) of rank n (with $q(x) = x{\bullet}x$) such that

- q is positive definite
- L is **even**, i.e. $q(L) \subset 2\mathbb{Z}$.

Nice lattices

(I) Let $\mathscr{L}_n$ be the set of **nice** lattices, i.e. unimodular quadratic lattices (L, q) of rank n (with $q(x) = x{\bullet}x$) such that

- q is positive definite
- L is **even**, i.e. $q(L) \subset 2\mathbb{Z}$.

(II) If $8 \mid n$ then one easily checks that $E_n \in \mathscr{L}_n$, where

$$E_n = \{x \in \mathbb{Z}^n | \, 2 \mid x_1 + ... + x_n\} + \mathbb{Z}{\bullet}(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$$

with the standard inner product. $\mathbb{Z}^n$ is never nice (sic!).

Theta functions

(I) Let $L \in \mathscr{L}_n$ and let $r_L(m)$ be the number of $x \in L$ for which $q(x) = x \bullet x = 2m$, a finite number since q is positive definite.

Theta functions

(I) Let $L \in \mathscr{L}_n$ and let $r_L(m)$ be the number of $x \in L$ for which $q(x) = x{\bullet}x = 2m$, a finite number since q is positive definite.

(II) The **theta function of** L (with $q = e^{2i\pi z}$, $z \in \mathscr{H}$)

$$\Theta_L(z) = \sum_{x \in L} q^{x \bullet x/2} = \sum_{m \geq 0} r_L(m) q^m$$

is a 1-periodic holomorphic function on $\mathscr{H}$, since $r_L(m) = O(m^{n/2})$.

Theta functions

(I) Here is a key result:

Theorem Suppose that $\mathscr{L}_n \neq \emptyset$ and let $L \in \mathscr{L}_n$. Then $8 \mid n$ and $\Theta_L \in M_{n/2}(\mathbb{SL}_2(\mathbb{Z}))$.

First, we prove that

$$\Theta_L(-1/z) = (-iz)^{n/2}\Theta_L(z).$$

It suffices to check it for $z = it$ with $t > 0$, i.e. we want

$$\sum_{x \in t^{-1/2}L} f(x) = t^{n/2} \sum_{x \in t^{1/2}L} f(x),$$

with $f(x) = e^{-\pi x \bullet x}$. A standard computation (reduce to dimension 1 via a ON-basis of $L \otimes \mathbb{R}$) shows that $\hat{f} = f$, where

$$\hat{f}(y) = \int_{L \otimes \mathbb{R}} e^{-2i\pi x \bullet y} f(x) dx.$$

Theta functions

(I) The trace formula (i.e. Poisson summation) applied to the compact quotient $(L \otimes \mathbb{R})/(t^{1/2}L)$ easily yields the result: $t^{-1/2}L$ is dual to $t^{1/2}L$ and the co-volume of $t^{1/2}L$ is $t^{n/2}$.

Theta functions

(I) The trace formula (i.e. Poisson summation) applied to the compact quotient $(L \otimes \mathbb{R})/(t^{1/2}L)$ easily yields the result: $t^{-1/2}L$ is dual to $t^{1/2}L$ and the co-volume of $t^{1/2}L$ is $t^{n/2}$.

(II) To finish the proof of the theorem, it suffices to check that $8 \mid n$. Replacing L by $L \oplus L$ or $L^{\oplus 4}$ we may assume that $4 \mid n$ and 8 does not divide n. Then by what we've just proved

$$\omega(z) = \Theta_L(z) dz^{n/4}$$

satisfies $S^*(\omega) = -\omega$ and $T^*(\omega) = \omega$ (where $S : z \to -1/z$ and $T : z \to z + 1$), thus $(ST)^*\omega = -\omega$, impossible since $(ST)^3 = 1$ and $\omega \neq 0$.

Applications

(I) Looking at constant terms we get, with $k = n/4$

$$\Theta_L - E_k \in S_{n/2} := S_{n/2}(\mathbb{SL}_2(\mathbb{Z})).$$

Hecke's trivial bound and the q-expansion of E_k give

$$r_L(m) = \frac{4k}{B_k}\sigma_{2k-1}(m) + O(m^k), \; k = n/4,$$

where the Bernoulli numbers are defined by

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k \geq 1} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}.$$

Applications

(I) Looking at constant terms we get, with $k = n/4$

$$\Theta_L - E_k \in S_{n/2} := S_{n/2}(\mathbb{SL}_2(\mathbb{Z})).$$

Hecke's trivial bound and the q-expansion of E_k give

$$r_L(m) = \frac{4k}{B_k}\sigma_{2k-1}(m) + O(m^k),\ \ k = n/4,$$

where the Bernoulli numbers are defined by

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k\geq 1}(-1)^{k+1}B_k\frac{x^{2k}}{(2k)!}.$$

(II) For $n = 8$ we have $S_4 = 0$ so $\Theta_L = E_2$ and $r_\Gamma(m) = 240\sigma_3(m)$. Mordell proved that any such L is isomorphic to E_8. For $n = 16$ we get $\Theta_L = E_4$ and $r_L(m) = 480\sigma_7(m)$.

Applications

(I) Witt proved that there are exactly two such L (up to isomorphism), namely $E_8 \oplus E_8$ and E_{16}. These give rise to non-isomorphic iso-spectral tori (Milnor).

Applications

(I) Witt proved that there are exactly two such L (up to isomorphism), namely $E_8 \oplus E_8$ and E_{16}. These give rise to non-isomorphic iso-spectral tori (Milnor).

(II) For $n = 24$ letting

$$\Delta = q\prod_n (1-q^n)^{24} = \sum \tau(n)q^n, \; E_6 = 1 + \frac{65520}{691}\sum \sigma_{11}(n)q^n,$$

$M_{12}(\mathbb{SL}_2(\mathbb{Z})) = \mathbb{C}E_6 \oplus \mathbb{C}\Delta$, thus $\exists c_L \in \mathbb{Q}$ such that

$$r_L(m) = \frac{65520}{691}\sigma_{11}(m) + c_L\tau(m).$$

Conway and Leech proved that there is a unique such L with $r_L(1) = 0$, the famous **Leech lattice**. Hence

$$r_{Leech}(m) = \frac{65520}{691}(\sigma_{11}(m) - \tau(m)), \quad \tau(m) \equiv \sigma_{11}(m) \pmod{691},$$

a famous **Ramanujan congruence**.

Counting nice lattices

(I) Let
$$X_n = \mathscr{L}_n / \simeq .$$
We've just seen that $X_n \neq \emptyset$ iff $8 \mid n$, and $|X_8| = 1, |X_{16}| = 2$. We'll see that X_n is finite. The next result is **much** deeper:

Theorem (Niemeier, King) We have $|X_{24}| = 24$ and $|X_{32}| > 10^9$.

$|X_n|$ has a very beautiful group-theoretic and adelic description, which requires some preliminary discussion.

Brief recollections on adèles

(I) Recall that the ring of adèles $\mathbb{A}$ is locally compact and $\mathbb{Q}$ is a co-compact lattice in it. An element of $\mathbb{A}$ is a family $(a_v)_v$ indexed by places v of $\mathbb{Q}$ (i.e. primes or ∞) with $a_v \in \mathbb{Q}_v$ and $a_v \in \mathbb{Z}_v$ for almost all v. We have

$$\mathbb{A} = \mathbb{R} \times \mathbb{A}_f, \; \mathbb{A}_f = \mathbb{Q} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}, \; \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p.$$

Brief recollections on adèles

(I) Recall that the ring of adèles $\mathbb{A}$ is locally compact and $\mathbb{Q}$ is a co-compact lattice in it. An element of $\mathbb{A}$ is a family $(a_v)_v$ indexed by places v of $\mathbb{Q}$ (i.e. primes or ∞) with $a_v \in \mathbb{Q}_v$ and $a_v \in \mathbb{Z}_v$ for almost all v. We have

$$\mathbb{A} = \mathbb{R} \times \mathbb{A}_f,\ \mathbb{A}_f = \mathbb{Q} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}},\ \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p.$$

(II) For any $\mathbb{Q}$-group G with a $\mathbb{Z}$-model $\mathscr{G}$, the topological group $G(\mathbb{A})$ consists of $(g_v)_v$ with $g_v \in G(\mathbb{Q}_v)$ and $g_v \in \mathscr{G}(\mathbb{Z}_v)$ for almost all v. The group $G(\mathbb{A})$ contains $G(\mathbb{Q})$ as a discrete subgroup. If G is semi-simple over $\mathbb{Q}$, then $G(\mathbb{Q})$ is a lattice in $G(\mathbb{A})$ (co-compact if and only if G is anisotropic over $\mathbb{Q}$), by the Borel and Borel-Harish-Chandra theorem.

Class numbers of algebraic groups

(I) Let $G \subset \mathbb{GL}_n(\mathbb{C})$ be a connected $\mathbb{Q}$-group. The next theorem is quite deep: applied to $G = (F \otimes_{\mathbb{Q}} \mathbb{C})^{\times}$, with F a number field, this gives the finiteness of the class number of F.

Theorem (Borel) The set $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}})$ is finite.

The **class number** of G is

$$\mathrm{cl}(G) = |G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}})|.$$

Be careful that it depends on the choice of the embedding $G \subset \mathbb{GL}_n(\mathbb{C})$ since $G(\hat{\mathbb{Z}})$ depends on that.

Class number of $\mathbb{GL}_n$ and $\mathbb{SL}_n$

(I) As an amuse-bouche, let's prove in two ways the following:

Theorem We have $\mathrm{cl}(\mathbb{GL}_n) = 1$ and $\mathrm{cl}(\mathbb{SL}_n) = 1$.

It suffices to check that $\mathrm{cl}(G) = 1$, where $G = \mathbb{SL}_n$, and this would follow from the density of $G(\mathbb{Q})$ in $G(\mathbb{A}_f)$: since $G(\hat{\mathbb{Z}})$ is open in $G(\mathbb{A}_f)$, any $G(\hat{\mathbb{Z}})$-orbit will intersect $G(\mathbb{Q})$ and thus $G(\mathbb{A}_f) = G(\mathbb{Q})G(\hat{\mathbb{Z}})$.

Class number of $\mathbb{GL}_n$ and $\mathbb{SL}_n$

(I) As an amuse-bouche, let's prove in two ways the following:

Theorem We have $\mathrm{cl}(\mathbb{GL}_n) = 1$ and $\mathrm{cl}(\mathbb{SL}_n) = 1$.

It suffices to check that $\mathrm{cl}(G) = 1$, where $G = \mathbb{SL}_n$, and this would follow from the density of $G(\mathbb{Q})$ in $G(\mathbb{A}_f)$: since $G(\hat{\mathbb{Z}})$ is open in $G(\mathbb{A}_f)$, any $G(\hat{\mathbb{Z}})$-orbit will intersect $G(\mathbb{Q})$ and thus $G(\mathbb{A}_f) = G(\mathbb{Q})G(\hat{\mathbb{Z}})$.

(II) Let H be the closure of $G(\mathbb{Q})$ in $G(\mathbb{A}_f)$. Then H contains $G(\mathbb{Q}_v)$ for any v: any $g \in G(\mathbb{Q}_v)$ is a product of elementary matrices, and $\mathbb{Q}$ is dense in $\mathbb{Q}_v$. Also H is closed in $G(\mathbb{A}_f)$, thus it contains $\prod_{v \in S} G(\mathbb{Q}_v) \times \prod_{v \notin S} G(\mathbb{Z}_v)$ for any finite set S. But then H contains the union of these over all S, which is $G(\mathbb{A}_f)$.

Local-global principle for lattices

(I) The second proof is based on the next key result. Let V be a finite dimensional $\mathbb{Q}$-vector space and let $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Let $\mathscr{L}(V)$ be the set of lattices in V. Define $\mathscr{L}(V_p)$ similarly. There is a natural map

$$\mathscr{L}(V) \to \prod_p \mathscr{L}(V_p), \; L \to (L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p)_p.$$

Theorem (Eichler) Fix a lattice $L_0 \subset V$. The above map induces a bijection between

$$\mathscr{L}(V) \simeq \prod_p{}' \mathscr{L}(V_p) :=$$

$$\{(L_p)_p \in \prod_p \mathscr{L}(V_p) | \, L_p = L_0 \otimes \mathbb{Z}_p \text{ for almost all } p\}.$$

Local-global principle for lattices

(I) Pick a basis of L_0 and identify it with $\mathbb{Z}^n$, and V with $\mathbb{Q}^n$. If $L \in \mathscr{L}(V)$, there is an integer $N \geq 1$ such that $\frac{1}{N}\mathbb{Z}^n \subset L \subset N\mathbb{Z}^n$. Thus $L_p = \mathbb{Z}_p^n$ inside $V_p = \mathbb{Q}_p^n$ for all p prime to N. Thus the map factors through $\prod_p' \mathscr{L}(V_p)$.

Local-global principle for lattices

(I) Pick a basis of L_0 and identify it with $\mathbb{Z}^n$, and V with $\mathbb{Q}^n$. If $L \in \mathscr{L}(V)$, there is an integer $N \geq 1$ such that $\frac{1}{N}\mathbb{Z}^n \subset L \subset N\mathbb{Z}^n$. Thus $L_p = \mathbb{Z}_p^n$ inside $V_p = \mathbb{Q}_p^n$ for all p prime to N. Thus the map factors through $\prod'_p \mathscr{L}(V_p)$.

(II) An easy exercise shows that for any lattice L we have $L = \cap_p L_p$ (with $L_p = L \otimes \mathbb{Z}_p$) inside $V \otimes \mathbb{A}_f$, giving injectivity. Using this recipe one also obtains an inverse of the map $L \to (L_p)_p$, namely $(L_p)_p \to \cap_p (L_p \cap V)$.

Local-global principle for lattices

(I) Take $V = \mathbb{Q}^n$ and $L_0 = \mathbb{Z}^n$. Then $\mathbb{GL}_n(\mathbb{A}_f) \simeq \prod'_p \mathbb{GL}(V_p)$ acts transitively on $\prod'_p \mathscr{L}(V_p)$, by $(g_p)_p \bullet (L_p)_p = (g_p(L_p))_p$, the stabiliser of $(\mathbb{Z}_p^n)_p$ being $\mathbb{GL}_n(\hat{\mathbb{Z}})$. Thus we obtain an identification

$$\mathbb{GL}_n(\mathbb{A}_f)/\mathbb{GL}_n(\hat{\mathbb{Z}}) \simeq \mathscr{L}(V) \simeq \mathbb{GL}_n(\mathbb{Q})/\mathbb{GL}_n(\mathbb{Z}),$$

giving $\mathbb{GL}_n(\mathbb{A}_f) = \mathbb{GL}_n(\mathbb{Q})\mathbb{GL}_n(\hat{\mathbb{Z}})$ and $\mathrm{cl}(\mathbb{GL}_n) = 1$.

Nice lattices and class numbers

(I) Fix n multiple of 8 and $L_0 \in \mathscr{L}_n$. Let $G = O(L_0)$ be the orthogonal group of L_0, a group defined over $\mathbb{Z}$, with $G(A)$ the automorphism group of the quadratic A-module $L_0 \otimes A$ for any A.

Theorem There is a natural bijection

$$X_n \to G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / G(\hat{\mathbb{Z}}),$$

thus $|X_n| = \mathrm{cl}(G)$.

In particular X_n is finite by Borel's theorem (we will see a different argument later on).

Nice lattices and class numbers

(I) A key input in the proof of the previous theorem is the following nontrivial result, using the classification of quadratic forms over $\mathbb{Z}_p$:

Theorem Any two lattices in $\mathscr{L}_n$ become isomorphic over $\mathbb{Z}_p$ for any prime p and thus (by the Hasse-Minkowski theorem) also over $\mathbb{Q}$.

Nice lattices and class numbers

(I) A key input in the proof of the previous theorem is the following nontrivial result, using the classification of quadratic forms over $\mathbb{Z}_p$:

Theorem Any two lattices in $\mathscr{L}_n$ become isomorphic over $\mathbb{Z}_p$ for any prime p and thus (by the Hasse-Minkowski theorem) also over $\mathbb{Q}$.

(II) Now pick $L \in \mathscr{L}_n$ and choose isomorphisms $\gamma : L \otimes \mathbb{Q} \to L_0 \otimes \mathbb{Q}$ and $\gamma_v : L \otimes \mathbb{Z}_v \to L_0 \otimes \mathbb{Z}_v$. Then $\gamma \circ \gamma_v^{-1} \in \mathrm{Aut}(L_0 \otimes \mathbb{Q}_v) = G(\mathbb{Q}_v)$ for all v, and they belong to $G(\mathbb{Z}_v)$ for almost all v.

Nice lattices and class numbers

(I) We obtain an element $g = (\gamma \circ \gamma_v^{-1})_v \in G(\mathbb{A})$. Changing γ multiplies g on the left by an element of $G(\mathbb{Q})$, and changing γ_v multiplies g on the right by an element of $G(\hat{\mathbb{Z}}_v)$ (resp. $G(\mathbb{R})$), thus the class of g in

$$G(\mathbb{Q})\backslash G(\mathbb{A})/G(\hat{\mathbb{Z}} \times \mathbb{R}) \simeq G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}})$$

is well-defined, and only depends on the isomorphism class of L, giving a map

$$X_n \to G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}}).$$

Nice lattices and class numbers

(I) We obtain an element $g = (\gamma \circ \gamma_v^{-1})_v \in G(\mathbb{A})$. Changing γ multiplies g on the left by an element of $G(\mathbb{Q})$, and changing γ_v multiplies g on the right by an element of $G(\hat{\mathbb{Z}}_v)$ (resp. $G(\mathbb{R})$), thus the class of g in

$$G(\mathbb{Q})\backslash G(\mathbb{A})/G(\hat{\mathbb{Z}} \times \mathbb{R}) \simeq G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}})$$

is well-defined, and only depends on the isomorphism class of L, giving a map

$$X_n \to G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}}).$$

(II) In the other direction, for any $g \in G(\mathbb{A}_f)$ we can use the action of $G(\mathbb{A}_f) \subset \mathbb{GL}(L_0 \otimes \mathbb{A}_f)$ on lattices in $L_0 \otimes \mathbb{Q}$ to get a lattice $L' = g(L_0) \subset L_0 \otimes \mathbb{Q}$. One easily checks that $L' \in \mathscr{L}_n$ and its isomorphism class depends only on the class of g in $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}})$. An easy exercise shows that these constructions are inverse to each other.

The mass formula

(I) Here is one of the most amazing formulae in mathematics. It gives the cardinality of X_n, "if we count correctly". Let $v(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ (volume of the sphere).

Theorem (Smith-Minkowski-Siegel) For any $n \in 8\mathbb{Z}_{>0}$

$$\sum_{L \in X_n} \frac{1}{|\mathrm{Aut}(L)|} = 2\zeta(n/2)\frac{\zeta(2)\zeta(4)\ldots\zeta(n-2)}{v(S^0)v(S^1)\ldots v(S^{n-1})}.$$

The theorem implies (exercise, using that $|\mathrm{Aut}(L)| \geq 2$) the existence of $c > 0$ such that for $8 \mid n$ we have $|X_n| > (cn)^{n^2}$. One can also write (exercise!)

$$\sum_{L \in X_n} \frac{1}{|\mathrm{Aut}(L)|} = 2^{-n}\frac{B_{n/4}}{n/4}\prod_{j=1}^{n/2-1}\frac{B_j}{j}.$$

The mass formula

(I) This formula is deeply related to adelic harmonic analysis! Pick a decomposition

$$G(\mathbb{A}_f) = \coprod_{i=1}^{h} G(\mathbb{Q}) g_i G(\hat{\mathbb{Z}})$$

and let $L_i \in X_n$ be the lattice corresponding to the class of g_i. We can compute the (finite) automorphism group of L_i by looking at those $g \in G(\mathbb{Q}) = \mathrm{Aut}(L_0 \otimes \mathbb{Q})$ which stabilise $L_i \otimes \mathbb{Z}_p$ for all p. We get

$$\mathrm{Aut}(L_i) = g_i K g_i^{-1} \cap G(\mathbb{Q}), \;\; K := G(\mathbb{R}) \times G(\hat{\mathbb{Z}}).$$

The mass formula

(I) This formula is deeply related to adelic harmonic analysis! Pick a decomposition

$$G(\mathbb{A}_f) = \coprod_{i=1}^{h} G(\mathbb{Q}) g_i G(\hat{\mathbb{Z}})$$

and let $L_i \in X_n$ be the lattice corresponding to the class of g_i. We can compute the (finite) automorphism group of L_i by looking at those $g \in G(\mathbb{Q}) = \mathrm{Aut}(L_0 \otimes \mathbb{Q})$ which stabilise $L_i \otimes \mathbb{Z}_p$ for all p. We get

$$\mathrm{Aut}(L_i) = g_i K g_i^{-1} \cap G(\mathbb{Q}), \;\; K := G(\mathbb{R}) \times G(\hat{\mathbb{Z}}).$$

(II) We have a decomposition (send $g\,\mathrm{Aut}(L_i)$ to the class of (g, g_i) for $g \in G(\mathbb{R})$)

$$G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\hat{\mathbb{Z}}) \simeq \coprod_{i=1}^{h} \mathrm{Aut}(L_i) \backslash G(\mathbb{R}).$$

The mass formula

(I) Picking compatible Haar measures μ on $G(\mathbb{A}_f)$, $G(\hat{\mathbb{Z}})$ and $G(\mathbb{R})$ we have (note that $G(\mathbb{R})$ is compact)

$$\frac{\mu(G(\mathbb{Q})\backslash G(\mathbb{A}))}{\mu(G(\hat{\mathbb{Z}}))} = \mu(G(\mathbb{R}))\sum_{i=1}^{h} \frac{1}{|\mathrm{Aut}(L_i)|},$$

thus

Theorem For any n multiple of 8 we have

$$\sum_{L\in X_n} \frac{1}{|\mathrm{Aut}(L)|} = \frac{\mu(G(\mathbb{Q})\backslash G(\mathbb{A}))}{\mu(G(\mathbb{R}) \times G(\hat{\mathbb{Z}}))}$$

for any Haar measure μ on $G(\mathbb{A})$.

The mass formula

(I) Picking compatible Haar measures μ on $G(\mathbb{A}_f)$, $G(\hat{\mathbb{Z}})$ and $G(\mathbb{R})$ we have (note that $G(\mathbb{R})$ is compact)

$$\frac{\mu(G(\mathbb{Q})\backslash G(\mathbb{A}))}{\mu(G(\hat{\mathbb{Z}}))} = \mu(G(\mathbb{R}))\sum_{i=1}^{h}\frac{1}{|\mathrm{Aut}(L_i)|},$$

thus

Theorem For any n multiple of 8 we have

$$\sum_{L\in X_n}\frac{1}{|\mathrm{Aut}(L)|} = \frac{\mu(G(\mathbb{Q})\backslash G(\mathbb{A}))}{\mu(G(\mathbb{R})\times G(\hat{\mathbb{Z}}))}$$

for any Haar measure μ on $G(\mathbb{A})$.

(II) We next explain, following Tamagawa and Weil, how to construct canonical Haar measures on semisimple $\mathbb{Q}$-groups (one can extend this to all $\mathbb{Q}$-groups, with extra work).

Measures and differential forms

(I) Let v be a place of $\mathbb{Q}$ and let X be a smooth variety of dimension n over $\mathbb{Q}_v$. The smoothness of X implies (via the implicit function theorem) that $X(\mathbb{Q}_v)$ has a natural structure of manifold, algebraic local coordinates at points of $X(\mathbb{Q}_v)$ giving rise to analytic charts around that point. Any algebraic differential n-form ω on X (defined over $\mathbb{Q}_v$) gives rise to a measure on $X(\mathbb{Q}_v)$, as follows.

Measures and differential forms

(I) Let v be a place of $\mathbb{Q}$ and let X be a smooth variety of dimension n over $\mathbb{Q}_v$. The smoothness of X implies (via the implicit function theorem) that $X(\mathbb{Q}_v)$ has a natural structure of manifold, algebraic local coordinates at points of $X(\mathbb{Q}_v)$ giving rise to analytic charts around that point. Any algebraic differential n-form ω on X (defined over $\mathbb{Q}_v$) gives rise to a measure on $X(\mathbb{Q}_v)$, as follows.

(II) Pick $x \in X(\mathbb{Q}_v)$ and local coordinates $t_1, \ldots, t_n$ near x (i.e. $t_1, \ldots, t_n$ generate the maximal ideal of the local ring at x). The t_i define a chart around x and we can express in this chart $\omega = g(t_1, \ldots, t_n) dt_1 \wedge \ldots \wedge dt_n$ for some power series g in $t_1, \ldots, t_n$, convergent on some ball around 0.

Measures and differential forms

(I) The measure $|g(t_1, ..., t_n)|_v dt_1...dt_n$ (where $dt_1...dt_n$ is the usual Haar measure on $\mathbb{Q}_v^n$, Lebesgue measure if $v = \infty$ and giving $\mathbb{Z}_v^n$ mass 1 if $v < \infty$) is independent of the choice of local coordinates (exchange coordinates one at a time and use Fubini to reduce to the case $n = 1$, which is elementary) and compatible with restriction to smaller open subsets around x. These measures glue to a measure $|\omega|$ on $X(\mathbb{Q}_v)$.

Theorem (Weil) If X has a smooth model $\mathscr{X}$ over $\mathbb{Z}_p$ and if ω is the restriction of a nowhere vanishing n-form on $\mathscr{X}$, then

$$\int_{\mathscr{X}(\mathbb{Z}_p)} |\omega| = \frac{|\mathscr{X}(\mathbb{F}_p)|}{p^{\dim X}}.$$

Measures and differential forms

(I) There is a natural surjective (by smoothness) reduction map

$$\mathrm{red} : \mathscr{X}(\mathbb{Z}_p) \to \mathscr{X}(\mathbb{F}_p)$$

and one checks (using a suitable form of Hensel's lemma and local inversion theorem) that local coordinates around $a \in \mathscr{X}(\mathbb{Z}_p)$ give rise to an analytic isomorphism

$$(p\mathbb{Z}_p)^{\dim X} \simeq \mathrm{red}^{-1}(\mathrm{red}(a)).$$

With respect to these local parameters $\omega = f(t_1, ..., t_n)dt_1 \wedge ... \wedge dt_n$ and $|f(t_1, ..., t_n)| = 1$ (since ω is nowhere vanishing mod p), thus

$$\int_{\mathrm{red}^{-1}(\mathrm{red}(a))} |\omega| = \int_{(p\mathbb{Z}_p)^{\dim X}} dt_1 ... dt_n = p^{-\dim X}.$$

The Tamagawa measure

(I) Let now G be a **semisimple** $\mathbb{Q}$-group of dimension n. The space Ω_G^{inv} of left-invariant nowhere-vanishing n-forms on G (defined over $\mathbb{Q}$) is one-dimensional over $\mathbb{Q}$. Any nonzero $\omega \in \Omega_G^{\mathrm{inv}}$ gives rise to measures $|\omega_v|$ on $G(\mathbb{Q}_v)$ for any place v of $\mathbb{Q}$, by the above recipe applied to G as a smooth $\mathbb{Q}_v$-variety. We want to define a measure on $G(\mathbb{A})$ by

$$|\omega| := \otimes_v |\omega_v|,$$

but one needs serious care in implementing this.

The Tamagawa measure

(I) Let now G be a **semisimple** $\mathbb{Q}$-group of dimension n. The space Ω_G^{inv} of left-invariant nowhere-vanishing n-forms on G (defined over $\mathbb{Q}$) is one-dimensional over $\mathbb{Q}$. Any nonzero $\omega \in \Omega_G^{\mathrm{inv}}$ gives rise to measures $|\omega_v|$ on $G(\mathbb{Q}_v)$ for any place v of $\mathbb{Q}$, by the above recipe applied to G as a smooth $\mathbb{Q}_v$-variety. We want to define a measure on $G(\mathbb{A})$ by

$$|\omega| := \otimes_v |\omega_v|,$$

but one needs serious care in implementing this.

(II) For some $N \geq 1$ G has a smooth model $\mathscr{G}$ over $\mathbb{Z}[1/N]$, and ω is induced by a nowhere-vanishing n-form on $\mathscr{G}$. By Weil's theorem we have

$$\prod_{\gcd(p,N)=1} |\omega_p|(\mathscr{G}(\mathbb{Z}_p)) = \prod_{\gcd(p,N)=1} \frac{|\mathscr{G}(\mathbb{F}_p)|}{p^n}.$$

The Tamagawa measure

(I) A deep theorem of Steinberg (crucially using that G is semisimple!) ensures that

$$\prod_{\gcd(p,N)=1} \frac{|\mathscr{G}(\mathbb{F}_p)|}{p^n} < \infty.$$

For instance $G = \mathbb{SL}_n$ we obtain

$$\prod_p \frac{\mathbb{SL}_n(\mathbb{F}_p)}{p^{n^2-1}} = \prod_p (1-p^{-n})(1-p^{1-n})...(1-p^{-2})$$

$$= \zeta(2)^{-1}\zeta(3)^{-1}...\zeta(n)^{-1}.$$

The Tamagawa measure

(I) A deep theorem of Steinberg (crucially using that G is semisimple!) ensures that

$$\prod_{\gcd(p,N)=1} \frac{|\mathscr{G}(\mathbb{F}_p)|}{p^n} < \infty.$$

For instance $G = \mathbb{SL}_n$ we obtain

$$\prod_p \frac{\mathbb{SL}_n(\mathbb{F}_p)}{p^{n^2-1}} = \prod_p (1-p^{-n})(1-p^{1-n})...(1-p^{-2})$$

$$= \zeta(2)^{-1}\zeta(3)^{-1}...\zeta(n)^{-1}.$$

(II) We can now define a measure on $G(\mathbb{A})$ as follows: for any M multiple of N pick the measure on $\prod_{\gcd(p,M)=1} \mathscr{G}(\mathbb{Z}_p)$ with total mass $\prod_{\gcd(p,M)=1} |\omega_p|(\mathscr{G}(\mathbb{Z}_p))$ and use the product measure on $\prod_{p|M} G(\mathbb{Q}_p) \times G(\mathbb{R})$.

The Tamagawa measure

(I) This gives us measures on $G(\mathbb{R}) \times \prod_{p|M} G(\mathbb{Q}_p) \times \prod_{\gcd(p,M)=1} G(\mathbb{Z}_p)$, which are compatible when increasing M, thus we get a measure on their union, which is $G(\mathbb{A})$. The result, the **Tamagawa measure** μ_G^{Tam}, is independent of any of the choices made, in particular of the choice of ω (since $|\lambda\omega_v| = |\lambda|_v|\omega_v|$ and $\prod_v |\lambda|_v = 1$ for $\lambda \in \mathbb{Q}^*$).

The Tamagawa measure

(I) This gives us measures on $G(\mathbb{R}) \times \prod_{p|M} G(\mathbb{Q}_p) \times \prod_{\gcd(p,M)=1} G(\mathbb{Z}_p)$, which are compatible when increasing M, thus we get a measure on their union, which is $G(\mathbb{A})$. The result, the **Tamagawa measure** μ_G^{Tam}, is independent of any of the choices made, in particular of the choice of ω (since $|\lambda\omega_v| = |\lambda|_v|\omega_v|$ and $\prod_v |\lambda|_v = 1$ for $\lambda \in \mathbb{Q}^*$).

(II) What really matters in practice is that for any continuous integrable functions f_v on $G(\mathbb{Q}_v)$ with $f_v = 1_{\mathscr{G}(\mathbb{Z}_v)}$ (for some model $\mathscr{G}$ over some $\mathbb{Z}[1/N]$) for almost all v, setting $f((g_v)_v) = \prod_v f_v(g_v)$ gives a continuous integrable function such that

$$\int_{G(\mathbb{A})} f(g)\mu_G^{\mathrm{Tam}}(g) = \prod_v \int_{G(\mathbb{Q}_v)} f_v(g_v)|\omega_v|(g_v).$$

The Tamagawa measure

(I) Since G is semi-simple, G has no algebraic characters (it is perfect!). Thus the (algebraic) action (right translation) of G on $\Omega^{\mathrm{inv}}(G)$ must be trivial and all such forms are left and right invariant. Thus μ^{Tam} is left and right invariant measure on $G(\mathbb{A}_f)$, which is thus unimodular (and so are all $G(\mathbb{Q}_v)$).

The Tamagawa measure

(I) Since G is semi-simple, G has no algebraic characters (it is perfect!). Thus the (algebraic) action (right translation) of G on $\Omega^{\mathrm{inv}}(G)$ must be trivial and all such forms are left and right invariant. Thus μ^{Tam} is left and right invariant measure on $G(\mathbb{A}_f)$, which is thus unimodular (and so are all $G(\mathbb{Q}_v)$).

(II) Since G is semi-simple, by the Borel-Harish-Chandra theorem we know that

$$\tau(G) := \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \mu_G^{Tam}(g)$$

is a real number (i.e. $G(\mathbb{Q})$ is a lattice in $G(\mathbb{A})$) called the **Tamagawa number of G**.

The Tamagawa measure

(I) The proof of the next theorem occupies a big chunk of Weil's book Adèles and algebraic groups:

Theorem (Tamagawa-Weil) We have $\tau(\mathbb{SL}_n) = 1$, $\tau(SO(q)) = 2$ for any non-degenerate quadratic form q over $\mathbb{Q}$ and $\tau(SL_1(D)) = 1$ for any division algebra D over $\mathbb{Q}$.

The equality $\tau(SO(q)) = 2$ is equivalent to the mass formula of Smith-Minkowski-Siegel when q is attached to an element of $\mathscr{L}_n$. For this one needs to compute the volume of $SO(q)(\hat{\mathbb{Z}} \times \mathbb{R})$, which reduces to computing $|SO(q)(\mathbb{F}_p)|$ and the volume of $SO(n)(\mathbb{R})$ (easily expressed inductively in terms of volumes of spheres).
Kottwitz proved (using deep work of Langlands and Arthur and many others) Weil's conjecture: $\tau(G) = 1$ for any connected, simply connected semi-simple group G over $\mathbb{Q}$.

Reduction theory for $\mathbb{GL}_n/\mathbb{Q}$

(I) Let $G = \mathbb{GL}_n(\mathbb{C})$, $K = O(n)$, A the subgroup of diagonal matrices with positive entries, N the group of upper triangular unipotent matrices in $G(\mathbb{R})$.

Reduction theory for $\mathbb{GL}_n/\mathbb{Q}$

(I) Let $G = \mathbb{GL}_n(\mathbb{C})$, $K = O(n)$, A the subgroup of diagonal matrices with positive entries, N the group of upper triangular unipotent matrices in $G(\mathbb{R})$.

(II) The Iwasawa decomposition (an easy exercise) says that multiplication gives a homeomorphism (even diffeo)

$$K \times A \times N \to G(\mathbb{R}).$$

(III) For $t > 0$ let

$$A_t := \{\mathrm{diag}(a_1, ..., a_n) \in A | \ \max(a_1/a_2, a_2/a_3, ...) \leq t\}$$

and for $u > 0$ let N_u be the subset of matrices in N whose off-diagonal entries belong to $[-u, u]$

Reduction theory for $\mathbb{GL}_n/\mathbb{Q}$

(I) Sets of the form

$$\Sigma_{t,u} = KA_tN_u$$

are called **Siegel sets** in $G(\mathbb{R})$.

Theorem (Hermite, Minkowski) We have

$$G(\mathbb{R}) = \Sigma_{2/\sqrt{3},1/2}G(\mathbb{Z}).$$

Using this, it is a simple (but excellent!) exercise to deduce the following basic result (already implicitly used...):

Theorem The set X_n is finite for all n.

Reduction theory for $\mathbb{GL}_n/\mathbb{Q}$

(I) Write $G_n = \mathbb{GL}_n(\mathbb{R}), \Gamma_n = \mathbb{GL}_n(\mathbb{Z})$ and $\Sigma_n = \Sigma_{2/\sqrt{3},1/2}$. Let $||\bullet||$ be the euclidean norm with respect to the canonical basis $e_1, \dots, e_n$ of $\mathbb{R}^n$. We will prove by induction on n that $\min_{x \in g\Gamma_n} ||xe_1||$ is reached in a point of Σ_n for any $g \in G_n$ (the min is reached since $g\Gamma_n(e_1) = g(\mathbb{Z}^n)$ is discrete), so that $g\Gamma_n$ intersects Σ_n.

Reduction theory for $\mathbb{GL}_n/\mathbb{Q}$

(I) Write $G_n = \mathbb{GL}_n(\mathbb{R}), \Gamma_n = \mathbb{GL}_n(\mathbb{Z})$ and $\Sigma_n = \Sigma_{2/\sqrt{3},1/2}$. Let $||\bullet||$ be the euclidean norm with respect to the canonical basis $e_1, ..., e_n$ of $\mathbb{R}^n$. We will prove by induction on n that $\min_{x \in g\Gamma_n} ||xe_1||$ is reached in a point of Σ_n for any $g \in G_n$ (the min is reached since $g\Gamma_n(e_1) = g(\mathbb{Z}^n)$ is discrete), so that $g\Gamma_n$ intersects Σ_n.

(II) Say the claim is proved for $n-1$ and let $g = kan$ such that $||ge_1|| = \min_{x \in g\Gamma_n} ||xe_1||$. First I claim that there is $\tilde{c} \in \Gamma_n$ such that $\tilde{c}e_1 = e_1$ and the Iwasawa decomposition of $g\tilde{c}$ is

$$g\tilde{c} = \tilde{k} \begin{pmatrix} a_1 & 0 \\ 0 & a'' \end{pmatrix} \tilde{n}, \ \ \tilde{n} \in N_{1/2}$$

$$a'' = \mathrm{diag}(a''_1, ..., a''_{n-1}), \ a''_i/a''_{i+1} \leq 2/\sqrt{3}.$$

Reduction theory for $\mathbb{GL}_n/\mathbb{Q}$

(I) Indeed, writing $a = \begin{pmatrix} a_1 & 0 \\ 0 & a' \end{pmatrix}$, $n = \begin{pmatrix} 1 & * \\ 0 & n' \end{pmatrix}$, by induction we can find $c' \in \Gamma_{n-1}$ such that $a'n'c' = k''a''n'' \in \Sigma_{n-1}$. A direct computation exhibits an identity of the form

$$g \begin{pmatrix} 1 & 0 \\ 0 & c' \end{pmatrix} = \tilde{k} \begin{pmatrix} a_1 & 0 \\ 0 & a'' \end{pmatrix} \begin{pmatrix} 1 & ** \\ 0 & n'' \end{pmatrix}.$$

But an easy induction shows that $N = N_{1/2}(\Gamma_n \cap N)$, so we can multiply $\begin{pmatrix} 1 & ** \\ 0 & n'' \end{pmatrix}$ by an element of $\Gamma_n \cap N$ to make it land in $N_{1/2}$.

Reduction theory for $\mathbb{GL}_n/\mathbb{Q}$

(I) Indeed, writing $a = \begin{pmatrix} a_1 & 0 \\ 0 & a' \end{pmatrix}$, $n = \begin{pmatrix} 1 & * \\ 0 & n' \end{pmatrix}$, by induction we can find $c' \in \Gamma_{n-1}$ such that $a'n'c' = k''a''n'' \in \Sigma_{n-1}$. A direct computation exhibits an identity of the form

$$g \begin{pmatrix} 1 & 0 \\ 0 & c' \end{pmatrix} = \tilde{k} \begin{pmatrix} a_1 & 0 \\ 0 & a'' \end{pmatrix} \begin{pmatrix} 1 & ** \\ 0 & n'' \end{pmatrix}.$$

But an easy induction shows that $N = N_{1/2}(\Gamma_n \cap N)$, so we can multiply $\begin{pmatrix} 1 & ** \\ 0 & n'' \end{pmatrix}$ by an element of $\Gamma_n \cap N$ to make it land in $N_{1/2}$.

(II) Back to the main business: since $\tilde{c}e_1 = e_1$, we have $||g\tilde{c}e_1|| = \min_{x \in g\tilde{c}\Gamma_n} ||xe_1||$, so we win by the following key lemma applied to $g\tilde{c}$:

Reduction theory for $\mathbb{GL}_n/\mathbb{Q}$

(I) Here's the key lemma:

Lemma Say $g = kan$ is such that $||ge_1|| = \min_{x \in g\Gamma_n} ||xe_1||$. There is $\bar{n} \in N_{1/2}$ such that $h := ka\bar{n} \in g\Gamma_n$ and $||ge_1|| = ||he_1||$. Moreover, $a_1/a_2 \leq 2/\sqrt{3}$.

The proof is simple. Pick $\bar{n} \in N_{1/2}$ such that $n \in \bar{n}(\Gamma_n \cap N)$ and set $h = ka\bar{n}$. Then $||ge_1|| = ||ae_1|| = a_1 = ||he_1||$. Next, if P is the matrix permuting e_1, e_2 and fixing $e_3, \ldots$, we have

$$a_1 = ||he_1|| \leq ||hPe_1|| = ||he_2|| = ||k(a_1\bar{n}_{12}e_1 + a_2e_2)||$$

$$= \sqrt{a_1^2\bar{n}_{12}^2 + a_2^2} \leq \sqrt{a_1^2/4 + a_2^2}$$

and we are done.

Proof of Mahler's compactness criterion

(I) Recall the statement:

Theorem (Mahler's compactness criterion) Let $M \subset \mathbb{GL}_n(\mathbb{R})$ be a subset such that for some $c > 0$ we have $\det(g) \geq c$ and $\inf_{x \in \mathbb{Z}^n \setminus \{0\}} ||g^{-1}x|| \geq c$ for $g \in M$. Then the image of M in $\mathbb{GL}_n(\mathbb{Z}) \backslash \mathbb{GL}_n(\mathbb{R})$ has compact closure.

Pick a sequence $g_j \in M$ and write $g_j^{-1} = k_j a_j n_j \gamma_j$ with $\gamma_j \in \mathbb{GL}_n(\mathbb{Z})$ and $k_j a_j n_j \in \Sigma_{2/\sqrt{3},1/2}$. It suffices to check that the a_j stay in a compact set, as then $\gamma_j g_j$ has a convergent sub-sequence. But if $a_j = \mathrm{diag}(a_j^1, a_j^2, \ldots)$ the condition on $\det g$ forces $a_j^1 \bullet a_j^2 \bullet \ldots$ to be bounded from above, so it suffices to check that all a_j^k stay away from 0. This follows from $a_j^1/a_j^2 \leq 2/\sqrt{3}, \ldots$ and

$$c \leq \inf_{x \in \mathbb{Z}^n \setminus \{0\}} ||g_j^{-1}x|| = \inf_x ||a_j n_j x|| \leq ||a_j n_j e_1|| = a_j^1.$$

$\mathbb{SL}_n(\mathbb{Z})$ is a lattice in $\mathbb{SL}_n(\mathbb{R})$

(I) This also gives a simple proof that $\mathbb{SL}_n(\mathbb{Z})$ is a lattice in $\mathbb{SL}_n(\mathbb{R})$. Let $\Sigma^1 = \Sigma_{2/\sqrt{3},1/2} \cap \mathbb{SL}_n(\mathbb{R})$, then one easily gets $\mathbb{SL}_n(\mathbb{R}) = \Sigma^1 \mathbb{SL}_n(\mathbb{Z})$, so it suffices to show that Σ^1 has finite Haar measure.

$\mathbb{SL}_n(\mathbb{Z})$ is a lattice in $\mathbb{SL}_n(\mathbb{R})$

(I) This also gives a simple proof that $\mathbb{SL}_n(\mathbb{Z})$ is a lattice in $\mathbb{SL}_n(\mathbb{R})$. Let $\Sigma^1 = \Sigma_{2/\sqrt{3},1/2} \cap \mathbb{SL}_n(\mathbb{R})$, then one easily gets $\mathbb{SL}_n(\mathbb{R}) = \Sigma^1 \mathbb{SL}_n(\mathbb{Z})$, so it suffices to show that Σ^1 has finite Haar measure.

(II) One has an Iwasawa decomposition $\mathbb{SL}_n(\mathbb{R}) = SO_n(\mathbb{R}) A_0 N$ with $A_0 = A \cap \mathbb{SL}_n(\mathbb{R})$ relative to which the Haar measure on $\mathbb{SL}_n(\mathbb{R})$ decomposes

$$dg = \prod_{i<j} \frac{a_i}{a_j} dk \bullet da \bullet dn.$$

Using this it's a simple exercise to check that Σ^1 has finite Haar measure.

The Tamagawa number of $\mathbb{SL}_n$

(I) We will sketch a rather geometric proof of $\tau(G) = 1$ for $G := \mathbb{SL}_n$. Let ω be the unique (up to a sign) invariant top-form on G, non-vanishing modulo any prime (exercise: write down one!). Since $\mathrm{cl}(G) = 1$, we have

$$G(\mathbb{Q})\backslash G(\mathbb{A})/G(\hat{\mathbb{Z}}) \simeq G(\mathbb{Z})\backslash G(\mathbb{R}).$$

Since

$$\mathrm{vol}(G(\hat{\mathbb{Z}})) = \prod_p |\omega_p|(G(\mathbb{Z}_p)) = \prod_p \frac{G(\mathbb{F}_p)}{p^{n^2-1}} = (\zeta(2)...\zeta(n))^{-1},$$

we are reduced to

$$\mathrm{vol}(D) := |\omega|_\infty(D) = \zeta(2)...\zeta(n)$$

for a fundamental domain D in $G(\mathbb{R})$ with respect to the action of $G(\mathbb{Z})$.

The Tamagawa number of $\mathbb{SL}_n$

(I) Consider the standard invariant top-form on $\mathbb{GL}_n$

$$\omega_{\mathrm{can}} = \frac{dx_{11} \wedge dx_{12} \wedge \ldots \wedge dx_{nn}}{\det(x_{ij})^n}.$$

Its pullback by the product map $m : \mathbb{SL}_n \times \mathbb{G}_m \to \mathbb{GL}_n$ is of the form $\alpha\omega \wedge \frac{dt}{t}$ (t the coordinate on $\mathbb{G}_m$) with α a constant. One can find α by looking at what's happening on tangent spaces at $(1, 1)$ and obtains $\alpha = \pm n$ and

$$m^*(dx_{11} \wedge dx_{12} \wedge \ldots \wedge dx_{nn}) = \pm n\omega \wedge t^{n^2-1}.$$

The Tamagawa number of $\mathbb{SL}_n$

(I) Consider the standard invariant top-form on $\mathbb{GL}_n$

$$\omega_{\mathrm{can}} = \frac{dx_{11} \wedge dx_{12} \wedge \ldots \wedge dx_{nn}}{\det(x_{ij})^n}.$$

Its pullback by the product map $m : \mathbb{SL}_n \times \mathbb{G}_m \to \mathbb{GL}_n$ is of the form $\alpha\omega \wedge \frac{dt}{t}$ (t the coordinate on $\mathbb{G}_m$) with α a constant. One can find α by looking at what's happening on tangent spaces at $(1,1)$ and obtains $\alpha = \pm n$ and

$$m^*(dx_{11} \wedge dx_{12} \wedge \ldots \wedge dx_{nn}) = \pm n\omega \wedge t^{n^2-1}.$$

(II) Thus letting $D_1 = m(D \times (0,1]) = \{tx|\, t \in (0,1], x \in D\}$ be the cone with section D, we have

$$\int_{D_1} |dx_{11} \wedge dx_{12} \wedge \ldots \wedge dx_{nn}| = \int_{D\times(0,1]} n|\omega|t^{n^2-1}dt = \frac{\mathrm{vol}(D)}{n}$$

and we need to show that

$$\mathrm{vol}(D_1) := \int_{D_1} |dx_{11} \wedge dx_{12} \wedge \ldots \wedge dx_{nn}| = \frac{\zeta(2)\ldots\zeta(n)}{n}.$$

The Tamagawa number of $\mathbb{SL}_n$

(I) For this we count lattice points in expanded versions of D, more precisely in $D_T := \{td \mid t \in (0, T], d \in D\}$ for $T \to \infty$. Note that $\mathrm{vol}(D_T) = T^{n^2}\mathrm{vol}(D_1)$, so we need to estimate

$$\mathrm{vol}(D_1) = \lim_{T\to\infty} \frac{\mathrm{vol}(D_T)}{T^{n^2}} = \lim_{T\to\infty} \frac{|D_T \cap M_n(\mathbb{Z})|}{T^{n^2}}$$

The Tamagawa number of $\mathbb{SL}_n$

(I) For this we count lattice points in expanded versions of D, more precisely in $D_T := \{td \mid t \in (0, T], d \in D\}$ for $T \to \infty$. Note that $\mathrm{vol}(D_T) = T^{n^2}\mathrm{vol}(D_1)$, so we need to estimate

$$\mathrm{vol}(D_1) = \lim_{T\to\infty} \frac{\mathrm{vol}(D_T)}{T^{n^2}} = \lim_{T\to\infty} \frac{|D_T \cap M_n(\mathbb{Z})|}{T^{n^2}}$$

(II) Since D_T is a fundamental domain for $\{X \in M_n(\mathbb{R}) \mid 0 < \det X \leq T^n\}$ modulo $G(\mathbb{Z})$, we obtain

$$\mathrm{vol}(D_1) = \lim_{T\to\infty} \frac{1}{T^{n^2}} \sum_{k=1}^{T^n} a_k,$$

where a_k is the number of matrices $X \in M_n(\mathbb{Z})$ with $\det X = k$, modulo $G(\mathbb{Z})$.

The Tamagawa number of $\mathbb{SL}_n$

(I) However, a_k is also the number of sub-lattices of $\mathbb{Z}^n$ of index k and a nice inductive argument based on elementary divisors shows that

$$\sum_{k\geq 1} \frac{a_k}{k^s} = \zeta(s)\zeta(s-1)...\zeta(s-n+1),$$

thus as $s \to 1$

$$\sum_{k\geq 1} \frac{a_k}{k^{s+n-1}} = \zeta(s)\zeta(s+1)...\zeta(s+n-1) \approx \zeta(2)...\zeta(n)/(s-1).$$

The Tamagawa number of $\mathbb{SL}_n$

(I) However, a_k is also the number of sub-lattices of $\mathbb{Z}^n$ of index k and a nice inductive argument based on elementary divisors shows that

$$\sum_{k\geq 1} \frac{a_k}{k^s} = \zeta(s)\zeta(s-1)...\zeta(s-n+1),$$

thus as $s \to 1$

$$\sum_{k\geq 1} \frac{a_k}{k^{s+n-1}} = \zeta(s)\zeta(s+1)...\zeta(s+n-1) \approx \zeta(2)...\zeta(n)/(s-1).$$

(II) Suitable Tauberian theorems then yield

$$\lim_{x\to\infty} \frac{1}{x^n} \sum_{k\leq x} a_k = \frac{\zeta(2)...\zeta(n)}{n}$$

and this finishes the proof.