Lecture 1: overview of the course

Gabriel Dospinescu

CNRS, ENS Lyon

Locally compact groups and their representations

(I) We want to study infinite dimensional representations of certain infinite, but very natural groups, such as $\mathbb{GL}_n(\mathbb{R})$, etc, and how they give rise to highly symmetric functions on these groups (automorphic forms).

Locally compact groups and their representations

(I) We want to study infinite dimensional representations of certain infinite, but very natural groups, such as $\mathbb{GL}_n(\mathbb{R})$, etc, and how they give rise to highly symmetric functions on these groups (automorphic forms).

(II) The theory is MUCH harder than that of finite dimensional representations of finite groups. All our representations will be on topological $\mathbb{C}$-vector spaces.

Locally compact groups and their representations

(I) We want to study infinite dimensional representations of certain infinite, but very natural groups, such as $\mathbb{GL}_n(\mathbb{R})$, etc, and how they give rise to highly symmetric functions on these groups (automorphic forms).

(II) The theory is MUCH harder than that of finite dimensional representations of finite groups. All our representations will be on topological $\mathbb{C}$-vector spaces.

(III) Let G be a locally compact group, for instance $\mathbb{R}, \mathbb{GL}_n(\mathbb{R})$, etc.

Locally compact groups and their representations

(I) We want to study infinite dimensional representations of certain infinite, but very natural groups, such as $\mathbb{GL}_n(\mathbb{R})$, etc, and how they give rise to highly symmetric functions on these groups (automorphic forms).

(II) The theory is MUCH harder than that of finite dimensional representations of finite groups. All our representations will be on topological $\mathbb{C}$-vector spaces.

(III) Let G be a locally compact group, for instance $\mathbb{R}, \mathbb{GL}_n(\mathbb{R})$, etc.

(IV) Even if we will focus on the case when G is a real Lie group in this course, it is important to deal with other groups, such as the ones above with $\mathbb{R}$ replaced by p-adic numbers or by adeles (if you know what these animals are!).

Locally compact groups and their representations

(I) Recall the famous but nontrivial result that on any such group G there is a unique (up to positive scalars) positive measure dg such that for any $f \in C_c(G)$ we have

$$\int_G f(x)dx = \int_G f(gx)dx$$

for all $g \in G$. We call such a measure a **left Haar measure** on G.

Locally compact groups and their representations

(I) Recall the famous but nontrivial result that on any such group G there is a unique (up to positive scalars) positive measure dg such that for any $f \in C_c(G)$ we have

$$\int_G f(x)dx = \int_G f(gx)dx$$

for all $g \in G$. We call such a measure a **left Haar measure** on G.

(II) We say that G is **unimodular** if there is (equivalently any) a left Haar measure that is also right-invariant. Most interesting groups are unimodular (eg. compact groups, $\mathbb{GL}_n(\mathbb{R})$, abelian groups), but there are non-unimodular natural groups, e.g. upper triangular matrices in $\mathbb{GL}_n(\mathbb{R})$.

Locally compact groups and their representations

(I) For simplicity we suppose from that on that G is unimodular. A very interesting representation of G is that of G on the space $L^2(G) := L^2(G, dg)$ of (equivalence classes of) square-integrable functions. A fundamental problem is that of "decomposing" this representation into irreducible representations of G.

Locally compact groups and their representations

(I) For simplicity we suppose from that on that G is unimodular. A very interesting representation of G is that of G on the space $L^2(G) := L^2(G, dg)$ of (equivalence classes of) square-integrable functions. A fundamental problem is that of "decomposing" this representation into irreducible representations of G.

(II) Things are tricky: if $G = \mathbb{R}$ one can show that $L^2(G)$ has NO irreducible sub-representation, which seems very strange!

Locally compact groups and their representations

(I) For simplicity we suppose from that on that G is unimodular. A very interesting representation of G is that of G on the space $L^2(G) := L^2(G, dg)$ of (equivalence classes of) square-integrable functions. A fundamental problem is that of "decomposing" this representation into irreducible representations of G.

(II) Things are tricky: if $G = \mathbb{R}$ one can show that $L^2(G)$ has NO irreducible sub-representation, which seems very strange!

(III) First, we need to make a bit more precise what kind of representations of G we want to study.

Locally compact groups and their representations

(I) For G as above we introduce the category

$$\mathrm{Rep}(G)$$

of **continuous representations** of G on Fréchet spaces over $\mathbb{C}$. An object of $\mathrm{Rep}(G)$ is thus a Fréchet space V with a linear action of G such that each $g \in G$ defines a continuous map $V \to V$ and for each $v \in V$ the orbit map $G \to V, g \to g.v$ is continuous (equivalently: the action map $G \times V \to V$ is continuous).

Locally compact groups and their representations

(I) For G as above we introduce the category

$$\mathrm{Rep}(G)$$

of **continuous representations** of G on Fréchet spaces over $\mathbb{C}$. An object of $\mathrm{Rep}(G)$ is thus a Fréchet space V with a linear action of G such that each $g \in G$ defines a continuous map $V \to V$ and for each $v \in V$ the orbit map $G \to V, g \to g.v$ is continuous (equivalently: the action map $G \times V \to V$ is continuous).

(II) Morphisms in $\mathrm{Rep}(G)$ are continuous $\mathbb{C}$-linear maps respecting the G-action. We write $\mathrm{Hom}_G(V, W)$ for the space of morphisms between V and W.

Locally compact groups and their representations

(I) For G as above we introduce the category

$$\mathrm{Rep}(G)$$

of **continuous representations** of G on Fréchet spaces over $\mathbb{C}$. An object of $\mathrm{Rep}(G)$ is thus a Fréchet space V with a linear action of G such that each $g \in G$ defines a continuous map $V \to V$ and for each $v \in V$ the orbit map $G \to V, g \to g.v$ is continuous (equivalently: the action map $G \times V \to V$ is continuous).

(II) Morphisms in $\mathrm{Rep}(G)$ are continuous $\mathbb{C}$-linear maps respecting the G-action. We write $\mathrm{Hom}_G(V, W)$ for the space of morphisms between V and W.

(III) Let $V \in \mathrm{Rep}(G)$. A **sub-representation** of V is a CLOSED subspace stable under the action of G. We say that V is **irreducible** if the only sub-representations are 0 and V.

Hilbert space representations

(I) In this lecture we will deal with the subcategory $\mathrm{Hilb}(G)$ of $Rep(G)$ consisting of V which are Hilbert spaces (separable, by convention). It contains the subcategory $\mathrm{Unit}(G)$ of **unitary representations**, i.e. those V on which G acts by unitary operators (i.e. isometries of V).

Hilbert space representations

(I) In this lecture we will deal with the subcategory $\mathrm{Hilb}(G)$ of *Rep*(G) consisting of V which are Hilbert spaces (separable, by convention). It contains the subcategory $\mathrm{Unit}(G)$ of **unitary representations**, i.e. those V on which G acts by unitary operators (i.e. isometries of V).

(II) The **unitary dual**

$$\hat{G}$$

is the set of equivalence classes of irreducible objects in $\mathrm{Unit}(G)$. So $\hat{G}$ classifies the irreducible unitary representations of G.

Hilbert space representations

(I) In this lecture we will deal with the subcategory $\mathrm{Hilb}(G)$ of $Rep(G)$ consisting of V which are Hilbert spaces (separable, by convention). It contains the subcategory $\mathrm{Unit}(G)$ of **unitary representations**, i.e. those V on which G acts by unitary operators (i.e. isometries of V).

(II) The **unitary dual**

$$\hat{G}$$

is the set of equivalence classes of irreducible objects in $\mathrm{Unit}(G)$. So $\hat{G}$ classifies the irreducible unitary representations of G.

(III) Describing $\hat{G}$ for a given G is in general a very hard problem, but for many groups this can be done, with a LOT of work.

Discrete series

(I) We can now reformulate the first fundamental problem: given G, describe $\hat{G}$ and decide which $\pi \in \hat{G}$ "contribute" (i.e. are subquotients) to $L^2(G)$.

Discrete series

(I) We can now reformulate the first fundamental problem: given G, describe $\hat{G}$ and decide which $\pi \in \hat{G}$ "contribute" (i.e. are subquotients) to $L^2(G)$.

(II) For instance, for which $\pi \in \hat{G}$ do we have $\mathrm{Hom}_G(\pi, L^2(G)) \neq 0$? Such π are called **square-integrable or discrete series representations** of G. They give rise to a subset $DS(G) \subset \hat{G}$.

Discrete series

(I) We can now reformulate the first fundamental problem: given G, describe $\hat{G}$ and decide which $\pi \in \hat{G}$ "contribute" (i.e. are subquotients) to $L^2(G)$.

(II) For instance, for which $\pi \in \hat{G}$ do we have $\mathrm{Hom}_G(\pi, L^2(G)) \neq 0$? Such π are called **square-integrable or discrete series representations** of G. They give rise to a subset $DS(G) \subset \hat{G}$.

(III) For instance $DS(\mathbb{R}) = 0$ and $DS(\mathbb{SL}_n(\mathbb{R})) = 0$ for $n \geq 3$ (this is highly nontrivial!). So $DS(G)$ can be a tiny part of $\hat{G}$.

Discrete series for $\mathbb{SL}_2(\mathbb{R})$

(I) Our **favorite group** in this course will be $G = \mathbb{SL}_2(\mathbb{R})$. For this group $DS(G)$ is infinite and can be described as follows.

Discrete series for $\mathbb{SL}_2(\mathbb{R})$

(I) Our **favorite group** in this course will be $G = \mathbb{SL}_2(\mathbb{R})$. For this group $DS(G)$ is infinite and can be described as follows.

(II) Let $K = SO_2(\mathbb{R})$, a maximal compact subgroup of G. To construct the discrete series of G we will use the analysis/geometry of the symmetric space G/K.

Discrete series for $\mathbb{SL}_2(\mathbb{R})$

(I) Our **favorite group** in this course will be $G = \mathbb{SL}_2(\mathbb{R})$. For this group $DS(G)$ is infinite and can be described as follows.

(II) Let $K = SO_2(\mathbb{R})$, a maximal compact subgroup of G. To construct the discrete series of G we will use the analysis/geometry of the symmetric space G/K.

(III) As a G-topological space this is the same as the **Poincaré upper half-plane**

$$\mathscr{H} = \{z = x + iy \in \mathbb{C} |\, y > 0\} = \{z \in \mathbb{C} |\, \mathrm{Im}(z) > 0\}.$$

Indeed, $\mathscr{H}$ has a natural action of G by

$$g.z = \frac{az+b}{cz+d}$$

if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the action is transitive (use upper triangular matrices in G) and the stabiliser of $i \in \mathscr{H}$ is K.

Discrete series for $\mathbb{SL}_2(\mathbb{R})$

(I) Now $\mathscr{H}$ is an open subspace of $\mathbb{C}$, so we can talk about holomorphic functions on $\mathscr{H}$. Let $\mathscr{O}(\mathscr{H})$ be the ring of such functions. It has a natural action of G, induced by that of G on $\mathscr{H}$.

Discrete series for $\mathbb{SL}_2(\mathbb{R})$

(I) Now $\mathscr{H}$ is an open subspace of $\mathbb{C}$, so we can talk about holomorphic functions on $\mathscr{H}$. Let $\mathscr{O}(\mathscr{H})$ be the ring of such functions. It has a natural action of G, induced by that of G on $\mathscr{H}$.

(II) We will "twist" this action by some cocycles to get our discrete series. Namely, consider the map

$$j : G \times \mathscr{H} \to \mathbb{C}, j(g, z) = cz + d, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

One easily checks that

$$j(gh, z) = j(g, hz)j(h, z).$$

Discrete series for $\mathbb{SL}_2(\mathbb{R})$

(I) Now $\mathscr{H}$ is an open subspace of $\mathbb{C}$, so we can talk about holomorphic functions on $\mathscr{H}$. Let $\mathscr{O}(\mathscr{H})$ be the ring of such functions. It has a natural action of G, induced by that of G on $\mathscr{H}$.

(II) We will "twist" this action by some cocycles to get our discrete series. Namely, consider the map

$$j : G \times \mathscr{H} \to \mathbb{C}, j(g,z) = cz + d, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

One easily checks that

$$j(gh,z) = j(g,hz)j(h,z).$$

(III) For an integer n we get an action of G on $\mathscr{O}(\mathscr{H})$ by

$$g.f(z) = j(g^{-1},z)^{-n} f(g^{-1}.z).$$

Discrete series for $\mathbb{SL}_2(\mathbb{R})$

(I) So explicitly

$$g^{-1}.f(z) = \frac{1}{(cz+d)^n} f(\frac{az+b}{cz+d})$$

if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Discrete series for $\mathbb{SL}_2(\mathbb{R})$

(I) So explicitly

$$g^{-1}.f(z) = \frac{1}{(cz+d)^n} f\left(\frac{az+b}{cz+d}\right)$$

if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(II) The **hyperbolic measure** $\frac{dxdy}{y^2}$ on $\mathscr{H}$ is G-invariant. Consider the space (below $z = x + iy$)

$$DS_n^+ = \{f \in \mathscr{O}(\mathscr{H}) \mid ||f||^2 := \int_{\mathscr{H}} |f(z)|^2 y^n \frac{dxdy}{y^2} < \infty\}.$$

Discrete series for $\mathbb{SL}_2(\mathbb{R})$

(I) So explicitly

$$g^{-1}.f(z) = \frac{1}{(cz+d)^n} f(\frac{az+b}{cz+d})$$

if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(II) The **hyperbolic measure** $\frac{dxdy}{y^2}$ on $\mathscr{H}$ is G-invariant. Consider the space (below $z = x + iy$)

$$DS_n^+ = \{f \in \mathscr{O}(\mathscr{H}) | \, ||f||^2 := \int_{\mathscr{H}} |f(z)|^2 y^n \frac{dxdy}{y^2} < \infty\}.$$

(III) It is a Hilbert space for the given norm (this is already not trivial) and one can prove that it is preserved by the previous action of G and is a unitary representation of G.

Discrete series for $\mathbb{SL}_2(\mathbb{R})$

(I) We then have the nontrivial:

Theorem We have $DS_n^+ \in DS(G)$ for $n \geq 2$.

(II) Keep the previous action (depending on $n \geq 2$) on DS_n^+ and define a new action $*$ on the same space by $g * f = \tilde{g}.f$ where $\tilde{g} = wgw^{-1}$ and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This gives rise to a new unitary representation of G, called DS_n^-.

Theorem The representations DS_n^+ and DS_m^- for $m, n \geq 2$ are pairwise non isomorphic and $DS(G)$ consists precisely of their isomorphism classes.

The unitary dual of $\mathbb{SL}_2(\mathbb{R})$: principal series

(I) Keep $G = \mathbb{SL}_2(\mathbb{R})$. We may wonder if the previous construction gives the whole $\hat{G}$. This is obviously false, since the trivial representation is in $\hat{G}$ and not in $DS(G)$. Actually there are many other representations in $\hat{G}$.

The unitary dual of $\mathbb{SL}_2(\mathbb{R})$: principal series

(I) Keep $G = \mathbb{SL}_2(\mathbb{R})$. We may wonder if the previous construction gives the whole $\hat{G}$. This is obviously false, since the trivial representation is in $\hat{G}$ and not in $DS(G)$. Actually there are many other representations in $\hat{G}$.

(II) G has a unique (up to isomorphism) irreducible representation of dimension n for each $n \geq 1$, but as long as $n \geq 2$ this is not in $\hat{G}$ (great exercise!).

The unitary dual of $\mathbb{SL}_2(\mathbb{R})$: principal series

(I) Keep $G = \mathbb{SL}_2(\mathbb{R})$. We may wonder if the previous construction gives the whole $\hat{G}$. This is obviously false, since the trivial representation is in $\hat{G}$ and not in $DS(G)$. Actually there are many other representations in $\hat{G}$.

(II) G has a unique (up to isomorphism) irreducible representation of dimension n for each $n \geq 1$, but as long as $n \geq 2$ this is not in $\hat{G}$ (great exercise!).

(III) The other elements of $\hat{G}$ are constructed by a process called (parabolic) **induction**. Namely, let B be the subgroup of upper-triangular matrices in G. Matrices in B are of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. Any (continuous) character $\chi : \mathbb{R}^* \to \mathbb{C}^*$ gives rise to a character of B, namely $\chi(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}) = \chi(a)$.

The unitary dual of $\mathbb{SL}_2(\mathbb{R})$: principal series

(I) Characters of $\mathbb{R}^*$ are well understood: they are $\chi(x) = sgn(x)^{\varepsilon}|x|^s$ with $\varepsilon \in \{0, 1\}$ and $s \in \mathbb{C}$ (exercise). Such χ is unitary if and only if $s \in i\mathbb{R}$.

The unitary dual of $\mathbb{SL}_2(\mathbb{R})$: principal series

(I) Characters of $\mathbb{R}^*$ are well understood: they are $\chi(x) = sgn(x)^{\varepsilon}|x|^s$ with $\varepsilon \in \{0, 1\}$ and $s \in \mathbb{C}$ (exercise). Such χ is unitary if and only if $s \in i\mathbb{R}$.

(II) Let $\delta : B \to \mathbb{R}_{>0}$ be the **modulus character** of B, sending $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ to a^2 (this is related to the non-unimodularity of B).

The unitary dual of $\mathbb{SL}_2(\mathbb{R})$: principal series

(I) Characters of $\mathbb{R}^*$ are well understood: they are $\chi(x) = sgn(x)^\varepsilon |x|^s$ with $\varepsilon \in \{0,1\}$ and $s \in \mathbb{C}$ (exercise). Such χ is unitary if and only if $s \in i\mathbb{R}$.

(II) Let $\delta : B \to \mathbb{R}_{>0}$ be the **modulus character** of B, sending $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ to a^2 (this is related to the non-unimodularity of B).

(III) Given a character χ, let $I(\chi)$ be the space of (measurable) functions $f : G \to \mathbb{C}$ such that $||f||^2 := \int_K |f(k)|^2 dk < \infty$ and $f(bg) = \chi(b)\delta(b)^{1/2} f(g)$ for $b \in B$ and $g \in G$. It has an action of G by right translation: $g.f(x) = f(xg)$ (this is not so obvious because of the L^2 condition on K...).

The unitary dual of $\mathbb{SL}_2(\mathbb{R})$: principal series

(I) The following result requires quite a bit of work:

Theorem If $\chi \in \widehat{\mathbb{R}^*}$ is a unitary character, then $I(\chi)$ is a unitary representation of G, and it is irreducible precisely when $\chi \neq \mathrm{sgn}$ (sending $x \to x/|x|$). Also $I(\mathrm{sgn})$ is the direct sum of two irreducible representations LDS^+, LDS^-, called **limits of discrete series**. Moreover, $I(\chi) \simeq I(\chi')$ if and only if $\chi' \in \{\chi, \chi^{-1}\}$, and $I(\chi)$ and $LDS^\pm$ are not discrete series.

Complementary series for $\mathbb{SL}_2(\mathbb{R})$

(I) We still haven't exhausted $\hat{G}$! There is another series of representations, the quite mysterious **complementary series**, indexed by $s \in (-1, 1)$ nonzero (but the reps. for s and for $-s$ are isomorphic, so we can restrict to $s \in (0, 1)$). These correspond to the characters $\chi_s(a) = |a|^s$, which are not unitary. Still, the miracle is that one can find a G-invariant inner product on $I(\chi_s)$ and by completion for this inner product we obtain an object $C(s) \in \hat{G}$, called the complementary series with parameter s.

The unitary dual for $\mathbb{SL}_2(\mathbb{R})$

(I) We then have the magical result:

Theorem (Bargmann) $\hat{G}$ consists precisely of the trivial representation, the discrete series $DS^{\pm n}$ (for $n \geq 2$), the limits of discrete series $LDS^{\pm}$, the principal series $I(\chi)$ for $\chi \in \widehat{\mathbb{R}^*} \setminus \{\mathrm{sgn}\}$ (up to $\chi \to \chi^{-1}$) and the complementary series $C(s)$ for $s \in (0, 1)$.

The unitary dual for $\mathbb{SL}_2(\mathbb{R})$

(I) We then have the magical result:

Theorem (Bargmann) $\hat{G}$ consists precisely of the trivial representation, the discrete series $DS^{\pm n}$ (for $n \geq 2$), the limits of discrete series $LDS^{\pm}$, the principal series $I(\chi)$ for $\chi \in \widehat{\mathbb{R}^*} \setminus \{\mathrm{sgn}\}$ (up to $\chi \to \chi^{-1}$) and the complementary series $C(s)$ for $s \in (0, 1)$.

(II) How on earth can one prove something like this? Answer: fine study of the actions of K and of the Lie algebra of G on the smooth vectors of representations of G. This study can be done in great generality and the remaining slides explain some of the key results.

Representations of compact groups

(I) Here $G = K$ is a compact group. Then $\mathrm{Rep}(K)$ looks very much like the category of finite dimensional representations of a finite group, thanks to the beautiful:

Theorem (Peter-Weyl) Any $V \in \hat{K}$ is finite dimensional and any $V \in \mathrm{Hilb}(K)$ is the Hilbert direct sum of irreducible representations. Moreover, as $K \times K$-representations

$$L^2(K) \simeq \widehat{\bigoplus_{\pi \in \hat{K}}} \pi \otimes_{\mathbb{C}} \pi^*.$$

Here $K \times K$ acts on $L^2(K)$ by $(k_1, k_2)f(x) = f(k_1^{-1} x k_2)$ and on $\pi \otimes \pi^*$ by $(k_1, k_2)(v \otimes l) = k_1 v \otimes k_2 l$. In particular in $\mathrm{Rep}(K)$

$$L^2(K) \simeq \widehat{\bigoplus_{\pi \in \hat{K}}} \pi^{\dim \pi}.$$

Representations of compact groups

(I) One can use the Peter-Weyl theorem to study $\mathrm{Rep}(G)$ even if G is not compact, by restricting $V \in \mathrm{Rep}(G)$ to K and looking at the **subspace of K-finite vectors** V_K of V, i.e. those $v \in V$ for which $\mathrm{Span}(K.v)$ is finite dimensional.

Representations of compact groups

(I) One can use the Peter-Weyl theorem to study $\mathrm{Rep}(G)$ even if G is not compact, by restricting $V \in \mathrm{Rep}(G)$ to K and looking at the **subspace of K-finite vectors** V_K of V, i.e. those $v \in V$ for which $\mathrm{Span}(K.v)$ is finite dimensional.

(II) Simple arguments show that

$$V_K \simeq \bigoplus_{\pi \in \hat{K}} V(\pi),$$

where $V(\pi) \simeq \pi \otimes \mathrm{Hom}_K(\pi, V)$ is the **π-isotypic component** of V, i.e. the sum of all subspaces of V stable under K and isomorphic to π as a K-rep.

Representations of compact groups

(I) One can use the Peter-Weyl theorem to study $\mathrm{Rep}(G)$ even if G is not compact, by restricting $V \in \mathrm{Rep}(G)$ to K and looking at the **subspace of K-finite vectors** V_K of V, i.e. those $v \in V$ for which $\mathrm{Span}(K.v)$ is finite dimensional.

(II) Simple arguments show that

$$V_K \simeq \bigoplus_{\pi \in \hat{K}} V(\pi),$$

where $V(\pi) \simeq \pi \otimes \mathrm{Hom}_K(\pi, V)$ is the **π-isotypic component** of V, i.e. the sum of all subspaces of V stable under K and isomorphic to π as a K-rep.

(III) Representations appearing in nature are **admissible**, i.e. $\dim V(\pi) < \infty$ for all $\pi \in \hat{K}$. Of course, this depends on the choice of K, but not in practice, as we will see.

Examples for $\mathbb{SL}_2(\mathbb{R})$

(I) Take $G = \mathbb{SL}_2(\mathbb{R})$. One can prove (cf. later lectures) that each $\pi \in \hat{G}$ is admissible with respect to the maximal compact subgroup K.

Examples for $\mathbb{SL}_2(\mathbb{R})$

(I) Take $G = \mathbb{SL}_2(\mathbb{R})$. One can prove (cf. later lectures) that each $\pi \in \hat{G}$ is admissible with respect to the maximal compact subgroup K.

(II) For instance, pick a unitary character $\chi(x) = (\mathrm{sgn}(x))^{\varepsilon}|x|^{it}$ with $\varepsilon \in \{0, 1\}$ and $t \in \mathbb{R}$. Suppose that $\chi \neq \mathrm{sgn}$.

Examples for $\mathbb{SL}_2(\mathbb{R})$

(I) Take $G = \mathbb{SL}_2(\mathbb{R})$. One can prove (cf. later lectures) that each $\pi \in \hat{G}$ is admissible with respect to the maximal compact subgroup K.

(II) For instance, pick a unitary character $\chi(x) = (\mathrm{sgn}(x))^{\varepsilon}|x|^{it}$ with $\varepsilon \in \{0, 1\}$ and $t \in \mathbb{R}$. Suppose that $\chi \neq \mathrm{sgn}$.

(III) Now $G = BK$ and $B \cap K = \{\pm 1\}$, thus restriction to K induces an isometric isomorphism

$$I(\chi) \simeq L^2_{\varepsilon}(K) := \{f \in L^2(K) |\, f(-k) = (-1)^{\varepsilon} f(k)\}.$$

This is K-equivariant, for the usual action of K on the RHS.

Examples for $\mathbb{SL}_2(\mathbb{R})$

(I) For $n \in \varepsilon + 2\mathbb{Z}$ consider the map $\varphi_n \in L^2_\varepsilon(K)$ defined by

$$\varphi_n(r_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}) = e^{2i\pi n\theta}.$$

Then Fourier analysis gives

$$L^2_\varepsilon(K) = \widehat{\bigoplus_{n\in\varepsilon+2\mathbb{Z}}} \mathbb{C}\varphi_n.$$

Examples for $\mathbb{SL}_2(\mathbb{R})$

(I) For $n \in \varepsilon + 2\mathbb{Z}$ consider the map $\varphi_n \in L^2_\varepsilon(K)$ defined by

$$\varphi_n(r_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}) = e^{2i\pi n\theta}.$$

Then Fourier analysis gives

$$L^2_\varepsilon(K) = \widehat{\bigoplus_{n\in\varepsilon+2\mathbb{Z}}} \mathbb{C}\varphi_n.$$

(II) For the discrete series DS_n^+ consider $f_n(z) = \frac{1}{(z+i)^n}$ and, for $j \geq 0$, $f_{n,j}(z) = f_n(z)(\frac{z-i}{z+i})^j$. One can prove that these functions are in DS_n^+ (excellent exercise) and the action of K on them is given by explicit characters:

$$r_\theta.f_n = e^{-in\theta} f_n, \;\; r_\theta.f_{n,j} = e^{-i(n+2j)\theta} f_{n,j}.$$

Examples for $\mathbb{SL}_2(\mathbb{R})$

(I) One can then check that we have isomorphisms of K-representations

$$DS_n^+ \simeq \widehat{\bigoplus_{j \geq 0}} (r_\theta \to e^{-i(n+2j)\theta}), \quad DS_n^- \simeq \widehat{\bigoplus_{j \geq 0}} (r_\theta \to e^{i(n+2j)\theta}).$$

This makes it clear that DS_n^+ and DS_m^- are never isomorphic.

Linearizing the action of G

(I) We suppose now that G is a closed subgroup of $\mathbb{GL}_n(\mathbb{R})$ for some $n \geq 1$. Then G has a natural structure of Lie group, and its Lie algebra

$$\mathfrak{g} := \{X \in M_n(\mathbb{R}) | \, e^{tX} \in G \, \forall t \in \mathbb{R}\}$$

is an $\mathbb{R}$-subspace of $M_n(\mathbb{R})$ stable under $(X, Y) \to [X, Y] := XY - YX$.

Linearizing the action of G

(I) We suppose now that G is a closed subgroup of $\mathbb{GL}_n(\mathbb{R})$ for some $n \geq 1$. Then G has a natural structure of Lie group, and its Lie algebra

$$\mathfrak{g} := \{X \in M_n(\mathbb{R}) |\, e^{tX} \in G \,\forall t \in \mathbb{R}\}$$

is an $\mathbb{R}$-subspace of $M_n(\mathbb{R})$ stable under $(X, Y) \to [X, Y] := XY - YX$.

(II) If G is connected, then e^X (for $X \in \mathfrak{g}$) generate G as abstract group, so we have a good control on G via $\mathfrak{g}$.

Linearizing the action of G

(I) We suppose now that G is a closed subgroup of $\mathbb{GL}_n(\mathbb{R})$ for some $n \geq 1$. Then G has a natural structure of Lie group, and its Lie algebra

$$\mathfrak{g} := \{X \in M_n(\mathbb{R}) | \, e^{tX} \in G \, \forall t \in \mathbb{R}\}$$

is an $\mathbb{R}$-subspace of $M_n(\mathbb{R})$ stable under $(X, Y) \to [X, Y] := XY - YX$.

(II) If G is connected, then e^X (for $X \in \mathfrak{g}$) generate G as abstract group, so we have a good control on G via $\mathfrak{g}$.

(III) If $V \in \mathrm{Rep}(G)$, $\mathfrak{g}$ has no reason to act on V, but we will see that it acts naturally on a dense subspace of V, that of smooth vectors.

Linearizing the action of G

(I) More precisely, say $v \in V$ is C^1 if for all $X \in \mathfrak{g}$ the limit

$$X.v := \lim_{t \to 0} \frac{e^{tX}.v - v}{t}$$

exists in V. By induction, say v is C^k if v is C^1 and $X.v$ is C^{k-1} for all $X \in \mathfrak{g}$.

Linearizing the action of G

(I) More precisely, say $v \in V$ is C^1 if for all $X \in \mathfrak{g}$ the limit

$$X.v := \lim_{t \to 0} \frac{e^{tX}.v - v}{t}$$

exists in V. By induction, say v is C^k if v is C^1 and $X.v$ is C^{k-1} for all $X \in \mathfrak{g}$.

(II) The subspace V^∞ of smooth vectors (i.e. those which are C^k for all k) turns out to be a dense subspace of V, stable under the action of G, and this action differentiates on it to an action of $\mathfrak{g}$, i.e. for all $X, Y \in \mathfrak{g}$ and $v \in V^\infty$ we have

$$X.(Y.v) - Y.(X.v) = [X, Y].v.$$

Linearizing the action of G

(I) More precisely, say $v \in V$ is C^1 if for all $X \in \mathfrak{g}$ the limit

$$X.v := \lim_{t \to 0} \frac{e^{tX}.v - v}{t}$$

exists in V. By induction, say v is C^k if v is C^1 and $X.v$ is C^{k-1} for all $X \in \mathfrak{g}$.

(II) The subspace V^∞ of smooth vectors (i.e. those which are C^k for all k) turns out to be a dense subspace of V, stable under the action of G, and this action differentiates on it to an action of $\mathfrak{g}$, i.e. for all $X, Y \in \mathfrak{g}$ and $v \in V^\infty$ we have

$$X.(Y.v) - Y.(X.v) = [X, Y].v.$$

(III) The passage $V \to V^\infty$ is bad in general: it can happen that V is irreducible and V^∞ is not a simple $\mathfrak{g}$-module, and it can happen that the closure in V of a $\mathfrak{g}$-stable subspace of V^∞ is not G-stable.

Linearizing the action of G

(I) Harish-Chandra fixed these annoying problems as follows:

- restrict the class of groups G to real reductive subgroups of $\mathbb{GL}_n(\mathbb{R})$. There are several possible definitions of a real reductive group, depending on one's taste/needs. For us it will be a subgroup defined by polynomial equations (in the matrix entries and the inverse of its determinant) with real coefficients and stable under transpose (e.g. $\mathbb{SL}_n(\mathbb{R})$).

- restrict to admissible representations V.

- replace V^∞ by the Harish-Chandra module of V

$$HC(V) := V_K \cap V^\infty.$$

Linearizing the action of G

(I) Harish-Chandra fixed these annoying problems as follows:

- restrict the class of groups G to real reductive subgroups of $\mathbb{GL}_n(\mathbb{R})$. There are several possible definitions of a real reductive group, depending on one's taste/needs. For us it will be a subgroup defined by polynomial equations (in the matrix entries and the inverse of its determinant) with real coefficients and stable under transpose (e.g. $\mathbb{SL}_n(\mathbb{R})$).

- restrict to admissible representations V.

- replace V^∞ by the Harish-Chandra module of V

$$HC(V) := V_K \cap V^\infty.$$

(II) $HC(V)$ is a purely algebraic object (no topology) having compatible representations of $\mathfrak{g}$ and K. Such gadgets are called $(\mathfrak{g}, K)$-modules.

The fundamental theorem

(I) The result that makes everything work for the study of (admissible) representations of real reductive groups is then:

Theorem (Harish-Chandra) Let G be a real reductive group and let K be a maximal compact subgroup.
1) Any $V \in \hat{G}$ is admissible.
2) If $V \in \mathrm{Rep}(G)$ is admissible, then $HC(V) = V_K$ (i.e. all K-finite vectors are smooth) and the sub-representations of V are in bijection with $\mathfrak{g}$ and K-stable subspaces of $HC(V)$ via the maps $W \to W_K$ and $X \to \bar{X}$ (closure in V).
In particular V is irreducible if and only if $HC(V) = V_K$ is a simple $(\mathfrak{g}, K)$-module.

An example: irreducibility of principal series for $\mathbb{SL}_2(\mathbb{R})$

(I) Let's illustrate the previous theorem for $G = \mathbb{SL}_2(\mathbb{R})$, a unitary character $\chi(x) = (\mathrm{sgn}(x))^{\varepsilon}|x|^{it}$ different from sgn. For simplicity let's suppose that $t \neq 0$.

An example: irreducibility of principal series for $\mathbb{SL}_2(\mathbb{R})$

(I) Let's illustrate the previous theorem for $G = \mathbb{SL}_2(\mathbb{R})$, a unitary character $\chi(x) = (\mathrm{sgn}(x))^{\varepsilon}|x|^{it}$ different from sgn. For simplicity let's suppose that $t \neq 0$.

(II) Recall that as K-modules

$$I(\chi) \simeq L^2_{\varepsilon}(K) = \widehat{\bigoplus_{n \in \varepsilon + 2\mathbb{Z}}} \mathbb{C}\varphi_n,$$

with $\varphi_n(r_\theta) = e^{2i\pi n\theta}$. These φ_n are K-finite, so correspond to some functions $f_n \in I(\chi)_K$ (which can be made quite explicit). Moreover

$$I(\chi)_K \simeq \bigoplus_{n \in \varepsilon + 2\mathbb{Z}} \mathbb{C}\varphi_n.$$

An example: irreducibility of principal series for $\mathbb{SL}_2(\mathbb{R})$

(I) The action of $\mathfrak{g}$ on $I(\chi)_K \subset I(\chi)^\infty$ extends (by $\mathbb{C}$-linearity) to an action of $\mathfrak{g}_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} \mathfrak{g}$ and for this action simple but painful computations show that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f_n = inf_n, \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} f_n = (it + 1 - n) f_{n-2},$$

$$\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} f_n = (it + 1 + n) f_{n+2}.$$

An example: irreducibility of principal series for $\mathbb{SL}_2(\mathbb{R})$

(I) The action of $\mathfrak{g}$ on $I(\chi)_K \subset I(\chi)^\infty$ extends (by $\mathbb{C}$-linearity) to an action of $\mathfrak{g}_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} \mathfrak{g}$ and for this action simple but painful computations show that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f_n = inf_n, \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} f_n = (it + 1 - n) f_{n-2},$$

$$\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} f_n = (it + 1 + n) f_{n+2}.$$

(II) Using this one can easily check that $I(\chi)_K$ is an irreducible $(\mathfrak{g}, K)$-module. Let $V \subset I(\chi)_K$ be a nonzero sub $(\mathfrak{g}, K)$-module. Using that $I(\chi)_K$ is a direct sum of characters of K, it follows that $f_n \in V$ for some n. The previous formulae then show that $f_{n\pm 2} \in V$ (the key point is that $it + 1 \pm n \neq 0$) and then by induction V contains all f_n and so $V = I(\chi)_K$.

Lattices

(I) We make now the situation more arithmetic by introducing besides the locally compact unimodular group G a lattice $\Gamma \subset G$, i.e. a discrete subgroup such that $\Gamma\backslash G$ has finite measure (with respect to the natural G-invariant measure induced by the Haar measure on G). The lattice is called co-compact if $\Gamma\backslash G$ is compact.

Lattices

(I) We make now the situation more arithmetic by introducing besides the locally compact unimodular group G a lattice $\Gamma \subset G$, i.e. a discrete subgroup such that $\Gamma\backslash G$ has finite measure (with respect to the natural G-invariant measure induced by the Haar measure on G). The lattice is called co-compact if $\Gamma\backslash G$ is compact.

(II) For instance, $\mathbb{Z}^n$ is a co-compact lattice in $\mathbb{R}^n$, while $\mathbb{SL}_n(\mathbb{Z})$ is a non-co-compact lattice in $\mathbb{SL}_n(\mathbb{R})$ (this is not at all obvious!).

Lattices

(I) We make now the situation more arithmetic by introducing besides the locally compact unimodular group G a lattice $\Gamma \subset G$, i.e. a discrete subgroup such that $\Gamma \backslash G$ has finite measure (with respect to the natural G-invariant measure induced by the Haar measure on G). The lattice is called co-compact if $\Gamma \backslash G$ is compact.

(II) For instance, $\mathbb{Z}^n$ is a co-compact lattice in $\mathbb{R}^n$, while $\mathbb{SL}_n(\mathbb{Z})$ is a non-co-compact lattice in $\mathbb{SL}_n(\mathbb{R})$ (this is not at all obvious!).

(III) Reductive groups defined over $\mathbb{Q}$ and having finite center give rise to lattices: $\Gamma := G(\mathbb{Z})$ is a lattice in $G(\mathbb{R})$. This is a very deep theorem of Borel and Harish-Chandra. There are also simple criteria to decide when this lattice is co-compact.

Lattices

(I) The previous remark uses notions we haven't introduced so far, so let me give a completely explicit and already nontrivial example. Let a, b be positive integers and consider the set S of $(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4$ such that $x_0^2 - ax_1^2 - bx_2^2 + abx_3^2 = 1$. Define

$$\Gamma = \{\begin{pmatrix} x_0 + x_1\sqrt{a} & x_2 + x_3\sqrt{a} \\ b(x_2 - x_3\sqrt{a}) & x_0 - x_1\sqrt{a} \end{pmatrix} \mid (x_0, x_1, x_2, x_3) \in S\}.$$

Lattices

(I) The previous remark uses notions we haven't introduced so far, so let me give a completely explicit and already nontrivial example. Let a, b be positive integers and consider the set S of $(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4$ such that $x_0^2 - ax_1^2 - bx_2^2 + abx_3^2 = 1$. Define

$$\Gamma = \{\begin{pmatrix} x_0 + x_1\sqrt{a} & x_2 + x_3\sqrt{a} \\ b(x_2 - x_3\sqrt{a}) & x_0 - x_1\sqrt{a} \end{pmatrix} \mid (x_0, x_1, x_2, x_3) \in S\}.$$

(II) Then Γ is a lattice in $\mathbb{SL}_2(\mathbb{R})$ and it is co-compact if and only if the equation $x^2 = ay^2 + bz^2$ has only the trivial solution $(0, 0, 0)$ in integers.

L^2 and lattices

(I) Let Γ be a lattice in a locally compact unimodular group G. Then $L^2(\Gamma\backslash G)$ with the natural action of G by right translation is a unitary representation of G.

L^2 and lattices

(I) Let Γ be a lattice in a locally compact unimodular group G. Then $L^2(\Gamma\backslash G)$ with the natural action of G by right translation is a unitary representation of G.

(II) Understanding this representation keeps people busy since many years and will do so for many years. It is related to many deep problems in number theory (Langlands program).

Theorem (Gelfand, Graev, Piatetski-Shapiro) If Γ is a co-compact lattice in a locally compact group G, then there is an isomorphism of unitary G-representations

$$L^2(\Gamma\backslash G) \simeq \widehat{\bigoplus_{\pi\in\hat{G}}} \pi \otimes \mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G))$$

and $\mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G))$ are finite dimensional.

The finiteness theorem

(I) The deepest result we will prove (more or less...) in this course is

Theorem (Harish-Chandra) If Γ is an "arithmetic lattice" in a semisimple group G defined over $\mathbb{Q}$, then for any $\pi \in \widehat{G(\mathbb{R})}$ we have

$$\dim \mathrm{Hom}_G(\pi, L^2(\Gamma \backslash G(\mathbb{R}))) < \infty.$$

Several objects above haven't been introduced so far (we will do this in future lectures), but you can safely suppose for now that $G = \mathbb{SL}_n$ in the above theorem and $\Gamma = \mathbb{SL}_n(\mathbb{Z})$, the theorem is already highly nontrivial in this case.

The finiteness theorem

(I) The theorem above is MUCH harder than the GGPS theorem and the proof uses automorphic forms. If Γ is not co-compact in $G(\mathbb{R})$, the theorem does not say as much about $L^2(\Gamma\backslash G(\mathbb{R}))$ as the GGPS theorem: we no longer have a discrete decomposition, there are "continuous parts" associated to Eisenstein series.

The finiteness theorem

(I) The theorem above is MUCH harder than the GGPS theorem and the proof uses automorphic forms. If Γ is not co-compact in $G(\mathbb{R})$, the theorem does not say as much about $L^2(\Gamma\backslash G(\mathbb{R}))$ as the GGPS theorem: we no longer have a discrete decomposition, there are "continuous parts" associated to Eisenstein series.

(II) It is an extremely difficult problem (already in the co-compact case) to decide for which $\pi \in \widehat{G(\mathbb{R})}$ we have

$$\mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G(\mathbb{R})) \neq 0.$$

The finiteness theorem

(I) The theorem above is MUCH harder than the GGPS theorem and the proof uses automorphic forms. If Γ is not co-compact in $G(\mathbb{R})$, the theorem does not say as much about $L^2(\Gamma\backslash G(\mathbb{R}))$ as the GGPS theorem: we no longer have a discrete decomposition, there are "continuous parts" associated to Eisenstein series.

(II) It is an extremely difficult problem (already in the co-compact case) to decide for which $\pi \in \widehat{G(\mathbb{R})}$ we have

$$\mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G(\mathbb{R})) \neq 0.$$

(III) Spaces of the form $\mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G(\mathbb{R}))$ are extremely interesting, they give rise to highly symmetric functions on the group, called automorphic forms.

Back to earth

(I) Let's take our favorite example $G = \mathbb{SL}_2(\mathbb{R})$. Given a lattice $\Gamma \subset G$, spectral theory and the previous results yield an orthogonal decomposition of G-representations

$$L^2(\Gamma \backslash G) = L^2_{\mathrm{disc}} \oplus L^2_{\mathrm{cont}},$$

where L^2_{disc} is the Hilbert direct sum of all irreducible sub-representations of $L^2(\Gamma \backslash G)$. Moreover we have an analogue of the GGPS decomposition for the space L^2_{disc}.

Back to earth

(I) Let's take our favorite example $G = \mathbb{SL}_2(\mathbb{R})$. Given a lattice $\Gamma \subset G$, spectral theory and the previous results yield an orthogonal decomposition of G-representations

$$L^2(\Gamma\backslash G) = L^2_{\mathrm{disc}} \oplus L^2_{\mathrm{cont}},$$

where L^2_{disc} is the Hilbert direct sum of all irreducible sub-representations of $L^2(\Gamma\backslash G)$. Moreover we have an analogue of the GGPS decomposition for the space L^2_{disc}.

(II) The continuous part L^2_{cont} is a direct integral of representations $\int_0^\infty \pi_{it}^{\oplus n_\Gamma} dt$, where π_{it} is the principal series corresponding to $\chi_t(x) = |x|^{it}$ and n_Γ is the number of "cusps" of Γ (so 0 if Γ is co-compact, 1 if $\Gamma = \mathbb{SL}_2(\mathbb{Z})$).

Back to earth

(I) Let's take our favorite example $G = \mathbb{SL}_2(\mathbb{R})$. Given a lattice $\Gamma \subset G$, spectral theory and the previous results yield an orthogonal decomposition of G-representations

$$L^2(\Gamma \backslash G) = L^2_{\mathrm{disc}} \oplus L^2_{\mathrm{cont}},$$

where L^2_{disc} is the Hilbert direct sum of all irreducible sub-representations of $L^2(\Gamma \backslash G)$. Moreover we have an analogue of the GGPS decomposition for the space L^2_{disc}.

(II) The continuous part L^2_{cont} is a direct integral of representations $\int_0^\infty \pi_{it}^{\oplus n_\Gamma} dt$, where π_{it} is the principal series corresponding to $\chi_t(x) = |x|^{it}$ and n_Γ is the number of "cusps" of Γ (so 0 if Γ is co-compact, 1 if $\Gamma = \mathbb{SL}_2(\mathbb{Z})$).

(III) Thus complementary series $C(s)$ can only occur in L^2_{disc}.

Back to earth

(I) Now let's take $\pi \in \hat{G}$ such that $\pi^K \neq 0$ (here π^K is the space of vectors fixed by K). For instance π could be π_{it} or $C(s)$. In all cases π^K is actually one-dimensional, generated by a smooth vector v (this follows from our explicit description of the restriction to K of irreducible unitary representations).

Back to earth

(I) Now let's take $\pi \in \hat{G}$ such that $\pi^K \neq 0$ (here π^K is the space of vectors fixed by K). For instance π could be π_{it} or $C(s)$. In all cases π^K is actually one-dimensional, generated by a smooth vector v (this follows from our explicit description of the restriction to K of irreducible unitary representations).

(II) Suppose that $\varphi : \pi \to L^2(\Gamma\backslash G)$ is a G-equivariant nonzero map. Then $f := \varphi(v)$ is a smooth vector in $L^2(\Gamma\backslash G)$ which is K-invariant, so gives rise to a function $f \in C^\infty(\Gamma\backslash G/K) \simeq C^\infty(\Gamma\backslash\mathscr{H})$, which is square-integrable.

Back to earth

(I) There is a hidden symmetry coming from higher-order differential operators on G and on $\mathscr{H}$. On $\mathscr{H}$ this is the hyperbolic Laplacian

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

a self-adjoint operator commuting with the action of G.

Back to earth

(I) There is a hidden symmetry coming from higher-order differential operators on G and on $\mathscr{H}$. On $\mathscr{H}$ this is the hyperbolic Laplacian

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

a self-adjoint operator commuting with the action of G.

(II) On G we have the Casimir operator. More precisely, $\mathfrak{g}$ acts on $C^\infty(G)$ by $X.f(g) = \lim_{t\to 0} \frac{f(ge^{tX})-f(g)}{t}$. Let's see any element of $\mathfrak{g}$ as en endomorphism of $C^\infty(G)$. If $X_1, X_2 \in \mathfrak{g}$ we write X_1X_2 for the composition of these endomorphisms of $C^\infty(G)$.

Back to earth

(I) Now consider the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of $\mathfrak{g}$ and the second-order differential operator (Casimir operator)

$$C = \frac{1}{2}h^2 + ef + fe.$$

Back to earth

(I) Now consider the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of $\mathfrak{g}$ and the second-order differential operator (Casimir operator)

$$C = \frac{1}{2}h^2 + ef + fe.$$

(II) One can prove that C commutes with the actions of G by left and right translation on $C^\infty(G)$. In particular C induces a G-invariant operator on $C^\infty(G)^K \simeq C^\infty(G/K) \simeq C^\infty(\mathscr{H})$. A painful computation shows that via these isomorphisms

$$C/2 = \Delta.$$

Back to earth

(I) Let's come back to our $v \in \pi^K$ and $\varphi : \pi \to L^2(\Gamma \backslash G)$ and $f = \varphi(v)$. A version of Schur's lemma shows that C acts on π by a scalar, which can be computed explicitly in terms of the classification. Now φ is G-equivariant, so C also acts on f by the same scalar. Since $C/2 = \Delta$, we deduce that $f \in C^\infty(\Gamma \backslash \mathscr{H})$ is an eigenfunction of Δ.

Back to earth

(I) Let's come back to our $v \in \pi^K$ and $\varphi : \pi \to L^2(\Gamma\backslash G)$ and $f = \varphi(v)$. A version of Schur's lemma shows that C acts on π by a scalar, which can be computed explicitly in terms of the classification. Now φ is G-equivariant, so C also acts on f by the same scalar. Since $C/2 = \Delta$, we deduce that $f \in C^\infty(\Gamma\backslash\mathscr{H})$ is an eigenfunction of Δ.

(II) The upshot is: elements of $\mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G))$ (with π spherical, i.e. $\pi^K \neq 0$) give rise to highly symmetric functions $f \in C^\infty(\Gamma\backslash\mathscr{H})$ square integrable solutions of the spectral problem $\Delta f = \lambda f$. Here $\lambda = \frac{1}{4} + t^2$ if $\pi = \pi_{it}$ ($t \in \mathbb{R}$) and $\lambda = \frac{1}{4} - s^2$ if $\pi = C(s)$ ($s \in (0,1)$).